94-95 PRICE TWO SHILLINGS.

RUDIMENTARY TREATISE

ON

LOGARITHMS

AND

MATHEMATICAL TABLES.

By HENRY LAW, C.E.

Cover

DOUBLE PART.

JOHN WEALE.

94-95

A RUDIMENTARY TREATISE ON LOGARITHMS.

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LONDON:
JOHN WEALE, 59, HIGH HOLBORN
1853.

PREFACE.

THE following little work is intended to have a twofold object—both to explain and illustrate the use and application of logarithms for the practical calculator, and to set forth and demonstrate their nature and properties for the Mathematical student. While, therefore, the Theoretical and Practical parts have been kept distinct, so that either might be separately studied or referred to, they have been so written with reference to each other, as to form but one connected treatise, which the student, who really wishes to become thoroughly acquainted with the subject, should entirely peruse. For, although a knowledge of their mathematical properties is not essential to a knowledge of their use, yet they are so intimately connected, that the acquirement of one greatly facilitates the acquirement of the other.

H. L.

OLD WINDSOR, *9th July*, 1850.

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A

RUDIMENTARY TREATISE

ON

LOGARITHMS.

CHAPTER I.

Explanation and Definitions of Logarithms.

THE word Logarithm is derived from two Greek words (λογος, *ratio*, and αριθμος, *number*), and signifies the *ratios of numbers.*

By the *ratio* of two numbers, or the *proportion* which one number bears to the other (the two terms being synonymous), is meant the *magnitude of the quotient* arising from the division of one number by the other. Thus, the ratio of 2 to 6 is expressed by $\frac{2}{6}$, and any other two numbers would be said to have the same ratio when the quotient arising from the division of one by the other was the same; so, $\frac{4}{12}$ being equal to $\frac{2}{6}$, 4 is said to have the same ratio to 12 that 2 has to 6. This is frequently written—

$$4 : 12 :: 2 : 6$$

and is read, as 4 is to 12 so is 2 to 6; it signifies nothing more than that the ratio of the two first numbers is the same as that of the two last, or that $\frac{4}{12} = \frac{2}{6}$.

A series of numbers is said to be in *continued proportion* when the ratio between each two consecutive numbers is the same, thus—

$$2, 6, 18, 54, 162,$$

are in continued proportion, because the ratios of 2 to 6, 6 to 18, 18 to 54, and 54 to 162, or $\frac{2}{6}$, $\frac{6}{18}$, $\frac{18}{54}$, $\frac{54}{162}$, are all equal. Now, the ratio of 2 to 18 is made up of the ratio of 2 to 6 and 6 to 18; but as these are equal, it is twice the ratio of 2 to 6; so, in like manner, the ratio of 2 to 54 is three times the ratio of 2 to 6; and the ratio of 2 to 162 is four times that of 2 to 6.

In order to examine some of the properties of a series of numbers in continued proportion, let us take the following, which is a more extensive series than the preceding:—

1,	2,	4,	8,	16,	32,	64,	128,	256,	512,	1024,	2048 . . .	(A),
0,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11 . . .	(B),

and let us place under each term in this series a number expressing how many times the ratio of 1 to 2 is contained in the ratio of 1 to that term; we see at once that under 1 we must place 0, and under 2 we must place 1; also, since the ratio of 1 to 4 is twice that of 1 to 2, under 4 we must place 2, and since the ratio of 1 to 8 is three times that of 1 to 2, under 8 we must place 3; and, proceeding in a similar manner, we shall obtain the numbers in the second line above. Now, the numbers which we have thus placed under the terms of the proportion are *logarithms* of those terms, *and are so called because they express the number of ratios of unity to the first term contained in the ratio of unity to the term under which they are placed.*

The numbers composing a series in continued proportion, similar to the above, will, on examination, be found to be derived from each other by the continual multiplication of the previous term by some constant number; thus, in the first series, 6 is derived from 2 by being multiplied by 3, and in like manner 18 is obtained from 6, and 54 from 18; so in the second series, the constant multiplier is 2, each term being derived from the preceding by multiplication by that number. A series of numbers thus obtained by the continual *multiplication* of its terms by a constant number is called a *geometrical series;* such is the series (A) above; while a series in which the terms are derived by the continual addition of a constant number is termed an *arithmetical series*, an example of which is afforded by the series (B) above.

Now, whatever the number may be by the continual multiplication of which the *geometrical series* is formed, if the series commences with unity, and under it is written the

arithmetical series formed by the continued addition of unity, commencing with the cypher, then will the numbers in the lower line express the number of ratios of unity to the first term, of which the ratio of unity to all the other terms is made up, and therefore they will be the logarithms of the numbers in the line above them. For example:—

1, 3, 9, 27, 81, 243 { are the *numbers* forming a *geometrical series;*

of which 0, 1, 2, 3, 4, 5 { are the *logarithms* forming an *arithmetical series.*

So, 1, 7, 49, 343, 2401 { are the *numbers* forming a *geometrical series;*

of which 0, 1, 2, 3, 4 { are the *logarithms* forming an *arithmetical series.*

And again, 1, 10, 100, 1000, 10000 { are the *numbers* forming a *geometrical series;*

of which 0, 1, 2, 3, 4 { are the *logarithms* forming an *arithmetical series.*

Now, from the very nature of a geometrical series, it follows, that its terms are all powers of the constant number by the multiplication of which they are produced, and therefore, in place of writing the numbers themselves, we might introduce the expression denoting the power, without actually performing the multiplication, and we should thus obtain for the three geometrical series above, writing them vertically instead of in horizontal lines*,

Nos.	Logs.	Nos.	Logs.	Nos.	Logs.
1 =	3^0	1 =	7^0	1 =	10^0
3 =	3^1	7 =	7^1	10 =	10^1
9 =	3^2	49 =	7^2	100 =	10^2
27 =	3^3	343 =	7^3	1000 =	10^3
81 =	3^4	2401 =	7^4	10000 =	10^4
243 =	3^5				

In these we perceive immediately that the numbers denoting the powers, or, as they are termed, the *indices* or *exponents* of the *powers*, are the same as the arithmetical series given above, and that they are therefore the logarithms of the numbers in the first columns. The constant number, of which the powers are successively taken, is termed the *root* or *radix*, and may have any value that we please assigned to it. Thus, we derive another definition of a logarithm, which may be de-

* It must be borne in mind that $a^0 = 1$, and $a^1 = a$, whatever the value of a may be.

scribed as *the index or exponent, to which a certain root or base must be involved, in order to be equal to the number of which it is the logarithm.* It is, therefore, evident that a given number may have any number of logarithms corresponding with it; or that the same logarithm may serve for several different numbers, according to the value assumed for the base or root to be involved, or what is the same thing, the common ratio of the geometrical progression*. Thus, in the examples above, the bases or common ratios are 3, 7, and 10.

We have, therefore, three distinctly different definitions, which may be given of logarithms, depending upon the particular way in which they are regarded, and we shall recapitulate these definitions, before proceeding farther, in order to insure their being thoroughly understood.

1. The logarithm of a given number is the number of ratios of some assumed constant number to unity, contained in the ratio of the given number to unity.

2. Logarithms are a series of *numbers* in *arithmetical progression*, answering to another series of numbers in *geometrical progression;* so taken that 0 in the first corresponds with 1 in the latter.

3. The logarithm of a number is the index or exponent of the power, to which a given constant base or root must be involved, to be equal to that number.

Whichever of these definitions may be adopted, the same general properties may be deduced as belonging to logarithms; we shall, however, in the following pages, consider them under the notion involved in the third definition, as the exponents of the powers of some constant root. And, in order to a more perfect conception of the subject, we shall first consider the properties of the exponents of powers generally.

CHAPTER II.

On the Exponents of Powers.

In algebra, the powers of a quantity, or the number of times that that quantity has been employed as a factor to produce a given quantity, are denoted by that number being written

* See page 13.

somewhat to the right and above the number or letter expressing the original quantity or root of the power. Thus, the square of 6 is written 6^2; the cube of x, x^3; and the fifth power of 12, 12^5. In the first example, as 6 enters twice as a factor, it is called the second power, and is denoted by 2 written over the 6; in the second example, as x enters three times as a factor, it is called the third power, and is denoted by a 3 written above the x; and in the last example, as 12 enters five times as a factor, it is termed the fifth power, and is written 12^5. The number thus placed over a number, to denote the power to which it is required to be raised, is termed the *index* or *exponent* of that power; as the former of these terms is sometimes employed in a different sense, to avoid ambiguity we shall use only the last. Thus, in the foregoing examples, 2, 3, and 5 are the exponents of the powers, to which the quantities 6, x, and 12 are to be respectively raised or involved.

Frequently letters are employed instead of numbers as exponents of powers; thus, x^a denotes that the quantity represented by x is to be raised to the power represented by a; and b^n, that the quantity b is to be raised to the power of n, or the nth power. The quantities, as x or b, in the foregoing examples, which have to be involved, or the powers of which are to be taken, are termed the *roots* or *bases*.

When it is desired to multiply any two powers of a quantity, a very little consideration will show that their product will be equal to a power of that same quantity, whose exponent is the sum of the two exponents of the powers to be multiplied. For, let us suppose the powers to be multiplied to be x^3 and x^2, then $x^3 = x.x.x$, and $x^2 = x.x$, therefore, $x^3 \times x^2 = x.x.x.x.x = x^5$, the exponent of which 5 is equal to $3 + 2$, the sum of the exponents of the two factors. And the converse of this rule holds good, for if it is required to divide a power of a given quantity by any other power of the same quantity, it is only necessary to subtract the exponent of the divisor from the exponent of the dividend to obtain the exponent of their quotient. Thus, let it be required to divide x^6 by x^2, we have $x^6 \div x^2 = \dfrac{x.x.x.x.x.x}{x.x} = x.x.x.x = x^4$, the exponent of which is equal to $6 - 2$.

Let us next examine the value of the power of a power; for instance, the square of x^3. In this case, we see at once that the square of x^3 is nothing more than x^3 multiplied by itself, and by

our former rule for the multiplication of powers, we have $x^3 \times x^3 = x^6$; if we had required the cube of x^3, it would have been $x^3 . x^3 . x^3 = x^9$, and for every higher power of x^3 we must add another 3 to the exponent; it is therefore obvious, that as the exponent of the original power has to be taken as many times as the exponent of the power to which it has to be raised, that the new exponent will be equal to the product of the other two; thus, in the above examples, $3 \times 2 = 6$, therefore $(x^3)^2 = x^6$, and $3 \times 3 = 9$, therefore $(x^3)^3 = x^9$. The converse of this rule also holds good, for if it is required to extract any root of a power, we have only to divide the exponent of the power by the exponent of the root, to obtain the exponent desired. Thus the square root of x^4 is x^2, because $4 \div 2 = 2$; and the square root of x^6 is x^3, because $6 \div 2 = 3$.

The four processes which we have here described are those which are of the most frequent occurrence, and as it is essential that they should be perfectly comprehended before entering on the use of logarithms, we shall recapitulate them in the form of rules.

1. The multiplication of the powers of any quantity is performed by the addition of their exponents; that is, $x^n \times x^m = x^{(n+m)}$.

2. The division of the power of a quantity by any other power of the same quantity, is performed by subtracting the exponent of the divisor from the exponent of the dividend; that is, $x^n \div x^m = x^{(n-m)}$.

3. The involution of any power of a quantity to some power is performed by multiplying its exponent by the exponent of the power to which it is to be raised; that is, the nth power of x^m is $x^{n.m}$.

4. The extraction of the root of any power is performed by dividing its exponent by the exponent of the root required; that is, the nth root of x^m is $x^{\frac{m}{n}}$.

In the last example we have an exponent differing from any which we have previously met with, namely, a *fractional* exponent; its use, however, in that example sufficiently explains its meaning, which is, that the quantity to which it is attached is to be raised to the power denoted by the numerator of the fraction, and is then to have the root extracted, which is denoted by the denominator of the same; or, the processes may be reversed, and the root first extracted, and then the power raised, since the order in which these operations are performed makes no difference in the final re-

sult. For example, let x above equal 4, m equal 3, and n equal 2; then $x^{\frac{m}{n}} = 4^{\frac{3}{2}}$, and if we take the cube of 4, which equals 64, and extract its square root, we obtain 8; or, if we first extract its square root we obtain 2, the cube of which is also equal to 8. And therefore we perceive that the final result is the same, whichever process is first performed.

In the example to the second rule, namely, $x^{(n-m)} = x^r$; if m is less than n, then the exponent r is a *positive* number, and x^r is termed a *direct* power of x; if, however, m exceeds n, then will r be a *negative* number, and in this case x^{-r} is termed an *inverse* power of x. In order to arrive at a correct idea of the value of an inverse power, we will take a direct power, and successively divide by its root, or subtract unity from its exponent, until we obtain a negative value; thus, let us start with x^3, then—

$$x^{3-1} = \frac{x^3}{x} = x^2$$

$$x^{2-1} = \frac{x^2}{x} = x^1 = x$$

$$x^{1-1} = \frac{x^1}{x} = x^0 = 1$$

$$x^{0-1} = \frac{x^0}{x} = \frac{1}{x} = x^{-1}$$

$$x^{-1-1} = \frac{1}{x} \div x = \frac{1}{x^2} = x^{-2}$$

$$x^{-2-1} = \frac{1}{x^2} \div x = \frac{1}{x^3} = x^{-3},$$

or, in more general terms, $x^{-n} = \frac{1}{x^n}$; that is to say, *the inverse power of any number is equal to unity divided by the direct power with an equal exponent.*

This last rule holds equally, when the exponent of the inverse power is a fraction, as it does when an integer; thus, $x^{-\frac{1}{n}}$ is equal to unity divided by the nth root of x, or to $\frac{1}{x^{\frac{1}{n}}}$.

We have then four different forms in which an exponent may be presented.

1. The *positive integral* exponent, as x^n, which denotes the

direct nth power of x, and is equal to the product arising from the multiplication of n factors, each equal to x.

2. The *negative integral* exponent, as x^{-n}, which denotes the *inverse* nth power of x, and is equal to unity divided n times by x.

3. The *positive fractional* exponent, as $x^{\frac{1}{n}}$, which denotes the *direct* nth root of x, and is equal to a quantity which, being multiplied n times by itself, shall be equal to x.

4. The *negative fractional* exponent, as $x^{-\frac{1}{n}}$, which denotes the *inverse* nth root of x, and is equal to unity divided by the *direct* nth root of x.

CHAPTER III.

Of various Systems of Logarithms.

We next proceed with the consideration of logarithms, under the view suggested in the third definition given of them in the first chapter, as the exponents of the powers of some constant number, taken as a base for the system. It was there stated that any value might be assumed for this base; two only have, however, been employed, namely, 2·7182818 and 10. The first was that adopted by Baron Napier, the inventor of logarithms, and was employed by him in the first system of logarithms, which was calculated by Briggs. The reason why such an intricate number was adopted, and not some simple integer, will be presently explained; it may be sufficient at present to state, that the system having this number for its base, being capable of being expressed more simply, and calculated more easily, than any other, was the reason of its adoption; from which circumstance, they have also been occasionally designated the *natural* system of logarithms; they are, however, more frequently termed *Napierean* or *Hyperbolic* logarithms. The latter term, derived from certain relations, found to exist between the logarithms of this system and the asymptotic spaces of the hyperbola, and which were believed to be peculiar to it, is very properly falling into disuse, since it is now known that the same property belongs to every system, the only difference being in the angle included between the asymptotes, which depends upon and varies with the value of the base; for in the system of which we are now speaking, in which the base equals 2·7182818, the

asymptotes are at right angles to each other; in the other system, having 10 for its base, they make an angle of 25°·7404.

It was soon perceived by Briggs, who had calculated the first table of Napierean logarithms, that several important advantages were possessed by a system of logarithms whose base was 10; he consequently proposed it to Baron Napier, by whom it was adopted, and it is now universally employed for the purposes of calculation; although the Napierean logarithms are always employed in the Differential and Integral Calculus, and the other higher branches of analysis.

The logarithms of any particular system are immediately reduced to those of any other system, by merely being multiplied by a constant number, whose value depends on the relative values of the bases of the two systems. Thus, the logarithms belonging to the common, or as it is sometimes termed *Briggean*, system, whose base is 10, are converted into the Napierean, whose base is 2·71828, by being multiplied by 2·3025851, while the Napierean are reduced to the Briggean by being divided by the same number, or, what is the same thing, being multiplied by its reciprocal 0·4342945. We shall not stop here to prove this, or to explain its reason, as it would involve considerations with which the student is supposed at present not to be acquainted; its demonstration will be given in a subsequent chapter*.

We shall now proceed to explain why the base of the system was altered from 2·71828 to 10, and to point out the advantages thereby attained.

In the tables below we have given the numbers whose logarithms are the first six integers in both systems, the left-hand table being taken to the Napierean base, and the right-hand to the base of 10.

Nos.		Base. Logs.	Nos.		Base. Logs.
1·00	=	$2{\cdot}71828^{0}$	1	=	10^{0}
2·72	=	$2{\cdot}71828^{1}$	10	=	10^{1}
7·39	=	$2{\cdot}71828^{2}$	100	=	10^{2}
20·09	=	$2{\cdot}71828^{3}$	1000	=	10^{3}
54·60	=	$2{\cdot}71828^{4}$	10000	=	10^{4}
148·41	=	$2{\cdot}71828^{5}$	100000	=	10^{5}
403·43	=	$2{\cdot}71828^{6}$	1000000	=	10^{6}

By reference to the right-hand table it will be seen that the common logarithm of 1 is 0, that of 10 is 1, and of 100 is

* See page 22.

2; it is requisite, however, for the purposes of general computation that we should know the logarithms of all the intermediate numbers included between these, as from 1 to 10, from 10 to 100, and so on. Now, since the logarithm of 1 is 0, and of 10 is 1, it follows that the logarithms of any intermediate numbers, greater than 1, but less than 10, must be some fraction, whose value lies between 0 and 1; and in like manner that, since the logarithm of 100 is 2, the logarithm of any intermediate number between 10 and 100, must have a value between 1 and 2. Interpolating, therefore, these fractional values of the logarithms of the intermediate numbers, we obtain the following series:—

Nos.	Logs.	Nos.	Logs.
1	$= 10^{0.00000}$	11	$= 10^{1.04139}$
2	$= 10^{0.30103}$	12	$= 10^{1.07918}$
3	$= 10^{0.47712}$	13	$= 10^{1.11394}$
4	$= 10^{0.60206}$	14	$= 10^{1.14613}$
5	$= 10^{0.69897}$	15	$= 10^{1.17609}$
6	$= 10^{0.77815}$	16	$= 10^{1.20412}$
7	$= 10^{0.84510}$	17	$= 10^{1.23045}$
8	$= 10^{0.90309}$	18	$= 10^{1.25527}$
9	$= 10^{0.95424}$	19	$= 10^{1.27875}$
10	$= 10^{1.00000}$	20	$= 10^{1.30103}$
		&c.	&c.

All numbers which are powers of 10, necessarily have integers for their logarithms, but the logarithms of all the intermediate numbers are compounded of an integer and a decimal fraction. The decimal portion is termed the *mantissa*, and the integer, which precedes it, is called the *index*, or *characteristic;* as, however, the former of these terms is frequently employed in a different sense, we shall here only use the latter.

In the foregoing Table, if we compare the logarithm of 2 with that of 20, we shall find that they only differ in the characteristic, the mantissa or decimal portion being identical in both; the reason of this will be very evident, if we consider that 20 is 2 multiplied by 10, and therefore that the logarithm of 20 is equal to the logarithm of 2, with that of 10 added to it, and, as the logarithm of 10 is an integral number, its addition only affects the value of the characteristic. In fact, the addition of 1 to the characteristic is multiplying the number which it represents by 10; in like manner, adding 2

to the characteristic, is multiplying the number by 100, and so on. Thus the logarithm

$$\begin{array}{rl} & \text{Of } 2 \text{ is } 0{\cdot}30103; \\ \text{Of } 2 \times 10 = & 20 \text{ is } 0{\cdot}30103 + 1 = 1{\cdot}30103; \\ \text{Of } 2 \times 100 = & 200 \text{ is } 0{\cdot}30103 + 2 = 2{\cdot}30103; \\ \text{Of } 2 \times 1000 = & 2000 \text{ is } 0{\cdot}30103 + 3 = 3{\cdot}30103. \end{array}$$

The mantissa, or the decimal portion of the logarithm, is always the same with the same figures, whether they are decimals or integers; it is only the *characteristic* which changes its value, with a change in the position of the decimal point. The value of the characteristic of the logarithm of a number is always one less than the number of integers in that number; thus, in the above example, when the number is 20 the characteristic is 1, when 200 it is 2, and when 2000 it is 3.

The characteristic, therefore, of the logarithms of all numbers

Equal to, or greater than	1,	but less than	10,	is	1,
,, ,,	10,	,,	100,	,,	2,
,, ,,	100,	,,	1000,	,,	3,
,, ,,	1000,	,,	10000,	,,	4,
	&c.		&c.		&c.

By way of further illustration, we will take the number 67854, and successively divide it by 10, examining the change thus produced in the value of the corresponding logarithms:

Nos.	Logs.
67854	= 4·831576
6785·4	= 3·831576
678·54 . . .	= 2·831576
67·854 . . .	= 1·831576
6·7854 . .	= 0·831576
·67854 . .	= 0·831576 − 1
·067854 . .	= 0·831576 − 2
·0067854 .	= 0·831576 − 3

We here perceive, as we have already stated, that, the figures remaining unaltered, no change takes place in the *mantissa* of the logarithm, but that as the number is successively divided by 10, the value of the *characteristic* is diminished by unity. We see further that, when the number is wholly a decimal fraction, the characteristic of its logarithm is *negative;* when the first figure after the decimal

point is a *significant* figure*, the characteristic of its logarithm is — 1, when a nought is interposed after the decimal point, so that the first significant figure is the second decimal figure, the characteristic is — 2, with two noughts it is — 3, and generally, the characteristic of the logarithm of a decimal fraction is a negative number, greater by unity than the number of noughts following the decimal point. Instead of writing, as we have above, 0·831576 — 3, the characteristic is placed to the left of the mantissa, with the negative sign *above* it, thus $\bar{3}$·831576. The negative sign is placed *above*, instead of *before* the characteristic, to denote that it is only the *characteristic* and not the mantissa that is negative. Thus, the characteristic of the logarithm of

·1	is	$\bar{1}$,
·01	„	$\bar{2}$,
·001	„	$\bar{3}$,
·0001	„	$\bar{4}$.
&c.		&c.

Since the characteristic of the logarithm of any number does not depend upon the value of the figures composing that number, and is so easily found by attention to the foregoing rules, it is usual to omit them altogether in the tables of logarithms, and only to give the mantissa or decimal portion.

It is only logarithms having 10 for their base which possess this important property, of having the same mantissa for the same figures, and this was the reason of that number being proposed by Briggs for the base of the common system of logarithms.

CHAPTER IV.

Mode of calculating Logarithms, and Demonstration of their Properties.

In the following Chapter the expressions, or formulæ employed for the calculation of logarithms, are mathematically deduced, and demonstrations are given of all the properties of logarithms referred to in any other portion of the work. By those not familiar with mathematical investigation, the present

* All the numerals are significant figures, with the exception of the cypher.

chapter may be omitted, as it is in nowise necessary to the proper understanding of the remainder of the work; the subject would, however, have been hardly complete without it, and it was considered that a rigid demonstration would be far more satisfactory to those by whom it could be followed, than a mere enunciation of the several propositions without any proof; and that, the reason of the several propositions and rules being understood, they would become much more firmly fixed in the memory, and their practical application and adaptation to peculiar cases rendered far more easy.

Definitions.

1. The *Power* of a number or quantity, is the product arising from the multiplication of that number, any number of times by itself.

2. The *Root*, or *base* of a power, is the number or quantity, by the continual multiplication of which by itself, that power is produced.

3. The *Exponent*, or *index of a power*, is the number of times that the root of that power enters into it, as a factor.

4. The *Exponent*, or *index of a root*, is the number of times that it must be employed as a factor, to produce a given power.

5. A *Logarithm* of a number, to any base, is the index or exponent of the power to which that base must be involved, to be equal to the number.

6. A *System of Logarithms*, is the collection of the logarithms of a series of numbers, taken to the same base.

Scholium. The logarithm of any number, as x, to any base b, is expressed by $\log_b x$; in like manner, the logarithm of the same number, to any other base, as ϵ, is written $\log_\epsilon x$.

7. A series of numbers is in *Arithmetical Progression*, when each number is derived from that which precedes it, by the *addition* of a constant number.

Scholium. Such a series is called an *Arithmetical Series*, and any one of the numbers composing it, a *term*.

8. The *Common Difference*, is the constant number, by the continual addition of which, an arithmetical series is formed.

9. A series of numbers is in *Geometrical Progression*, when each number is derived from that which precedes it, by the *multiplication* by a constant number.

Scholium. Such a series is called a *Geometrical Series*.

10. The *Common Ratio*, is the constant number, by the

continual multiplication by which, a geometrical series is formed.

SCHOLIUM. In investigations similar to the following, the term *coefficient* is employed in a somewhat extended signification, to mean any quantity or expression (however complicated) by which the quantity, more immediately under consideration, is multiplied.

Thus, in the expression,

$$\frac{\frac{r}{\lambda}\left(\frac{r}{\lambda}-1\right)}{2} z^2 + \frac{\frac{r}{\lambda}\left(\frac{r}{\lambda}-1\right)\cdot\left(\frac{r}{\lambda}-2\right)}{2.3} z^3,$$

the quantities $\dfrac{\frac{r}{\lambda}\left(\frac{r}{\lambda}-1\right)}{2}$ and $\dfrac{\frac{r}{\lambda}\left(\frac{r}{\lambda}-1\right)\cdot\left(\frac{r}{\lambda}-2\right)}{2.3}$.

are looked upon as the coefficients of z^2 and z^3 respectively.

11. The *Characteristic* of a Logarithm, is the integral number, to the left of the decimal point.

12. The *Mantissa* of a Logarithm, is the decimal number, to the right of the decimal point.

13. A *Significant Figure* is every figure but a cypher; the *cypher* signifying no actual quantity, but being employed only to determine the place of the other figures.

PROPOSITION A.

THEOREM. *In an equation of the form*

$$\begin{gathered} A + Bx + Cx^2 + Dx^3 + \dots\dots\dots + \&c. = \\ a + bx + cx^2 + dx^3 + \dots\dots\dots + \&c. \end{gathered}$$

the coefficient of any power of x on one side of the equation, is equal to the coefficient of the like power of x on the other side; that is, $A = a$, $B = b$, $C = c$, &c.

Because, in the above expression, the values of the coefficients are perfectly independent of the value of x, therefore, we may assume x to have any value we please, without destroying the equation.

Let, therefore, $x = 0$, the equation then becomes

$$A = a.$$

Now, since A and a are equal, we may remove them from the original equation, which then becomes

$$\begin{gathered} Bx + Cx^2 + Dx^3 + \dots + \&c. = \\ bx + cx^2 + dx^3 + \dots + \&c. \end{gathered}$$

Dividing both sides by x, we obtain

$$\mathrm{B} + \mathrm{C}\,x + \mathrm{D}\,x^2 + \ldots + \&c. =$$
$$b + c\,x + d\,x^2 + \ldots + \&c.$$

And again, assuming $x = 0$, we have

$$\mathrm{B} = b.$$

And in like manner, it may be shown that $\mathrm{C} = c$, $\mathrm{D} = d$, &c.

SCHOLIUM. 1. The above Theorem is true, whatever signs the terms of the equation may be affected with, provided only, that the terms involving like powers of x, on the opposite sides, are affected with like signs. Thus, it is true if

$$\mathrm{A} - \mathrm{B}\,x + \mathrm{C}\,x^2 - \mathrm{D}\,x^3 + \ldots - \&c. =$$
$$a - b\,x + c\,x^2 - d\,x^3 + \ldots - \&c.$$

or if,

$$-\,\mathrm{A} + \mathrm{B}\,x - \mathrm{C}\,x^2 + \mathrm{D}\,x^3 - \ldots + \&c. =$$
$$-\,a + b\,x - c\,x^2 + d\,x^3 - \ldots + \&c.$$

2. This Theorem also holds good when more complicated functions of x take the place of x, x^2, x^3, &c., provided only, that the same functions occur in the same order on opposite sides; as for example, if

$$\mathrm{A} + \mathrm{B}\,x + \mathrm{C}\,x^2 + \mathrm{D}\,x\,y + \mathrm{E}\,y^2 + \mathrm{F}\,x^3 + \ldots + \&c. =$$
$$a + b\,x + c\,x^2 + d\,x\,y + e\,y^2 + f\,x^3 + \ldots + \&c.$$

PROPOSITION B.

PROBLEM. *To expand b^{λ} in terms of λ.*

For b, substitute $(1 + y)$, then $b^{\lambda} = (1 + y)^{\lambda}$; expanding this expression by the Binomial Theorem*, it becomes

$$b^{\lambda} = 1 + \lambda\,y + \lambda\,\frac{(\lambda - 1)}{2}\,y^2 + \lambda\,\frac{(\lambda - 1)\,.\,(\lambda - 2)}{2\,.\,3}\,y^3 +$$
$$\lambda\,\frac{(\lambda - 1)\,.\,(\lambda - 2)\,.\,(\lambda - 3)}{2\,.\,3\,.\,4}\,y^4 + \ldots\ldots + \&c.$$

$$= 1 + \lambda\,y + \frac{\lambda^2 - \lambda}{2}\,y^2 + \frac{\lambda^3 - 3\lambda^2 + 2\,\lambda}{2\,.\,3}\,y^3 +$$
$$\frac{\lambda^4 - 6\,\lambda^3 + 11\,\lambda^2 - 6\,\lambda}{2\,.\,3\,.\,4}\,y^4 + \ldots\ldots + \&c.$$

* For a demonstration of the Binomial Theorem, see the "Elements of Algebra," p. 148, by Mr. Haddon.

$$= 1 + \lambda y + \left(\frac{\lambda^2}{2} - \frac{\lambda}{2}\right) y^2 + \left(\frac{\lambda^3}{6} - \frac{3\lambda^2}{6} + \frac{2\lambda}{6}\right) y^3 +$$

$$\left(\frac{\lambda^4}{24} - \frac{6\lambda^3}{24} + \frac{11\lambda^2}{24} - \frac{6\lambda}{24}\right) y^4 + \ldots\ldots + \&c.$$

Arranging this last expression according to the powers of λ, we have

$$b^\lambda = 1 + \lambda \left\{ y - \frac{1}{2} y^2 + \frac{1}{3} y^3 - \frac{1}{4} y^4 + \ldots - \&c. \right\}$$

$$+ \lambda^2 \left\{ \frac{1}{2} y^2 - \frac{1}{2} y^3 + \frac{11}{24} y^4 - \ldots + \&c. \right\}$$

$$+ \lambda^3 \left\{ \frac{1}{6} y^3 - \frac{1}{4} y^4 + \ldots - \&c. \right\}$$

Or, if we put

$$A = \left\{ y - \frac{1}{2} y^2 + \frac{1}{3} y^3 - \frac{1}{4} y^4 + \ldots - \&c. \right\}$$

$$B = \left\{ \frac{1}{2} y^2 - \frac{1}{2} y^3 + \frac{11}{24} y^4 - \ldots + \&c. \right\}$$

$$C = \left\{ \frac{1}{6} y^3 - \frac{1}{4} y^4 + \ldots - \&c. \right\}$$

we have

$$b^\lambda = 1 + A\lambda + B\lambda^2 + C\lambda^3 + \ldots + \&c. \ \ldots\ldots \ [1.]$$

Now, in order to obtain the values of the coefficients B, C, &c., in terms of A, let us put z for $A\lambda + B\lambda^2 + C\lambda^3 + \&c.$, then the above expression becomes

$$b^\lambda = 1 + z.$$

Extracting the root on both sides, we have,

$$b = (1 + z)^{\frac{1}{\lambda}},$$

and raising them to the power of r, it becomes

$$b^r = (1 + z)^{\frac{r}{\lambda}}.$$

Expanding by the Binomial Theorem, we have

$$b^r = 1 + \frac{r}{\lambda} z + \frac{\frac{r}{\lambda}\left(\frac{r}{\lambda} - 1\right)}{2} z^2 + \frac{\frac{r}{\lambda}\left(\frac{r}{\lambda} - 1\right) \cdot \left(\frac{r}{\lambda} - 2\right)}{2 . 3} z^3 +$$

$$\frac{\frac{r}{\lambda}\left(\frac{r}{\lambda} - 1\right) \cdot \left(\frac{r}{\lambda} - 2\right) \cdot \left(\frac{r}{\lambda} - 3\right)}{2 . 3 . 4} z^4 + \ldots\ldots + \&c.$$

Substituting, in this expression, $A\lambda + B\lambda^2 + C\lambda^3 + \&c.$, for z, it becomes

$$b^r = 1 + \frac{r}{\lambda}(A\lambda + B\lambda^2 + C\lambda^3 + \&c.) +$$

$$\frac{\frac{r}{\lambda}\left(\frac{r}{\lambda} - 1\right)}{2}(A\lambda + B\lambda^2 + C\lambda^3 + \&c.)^2 +$$

$$\frac{\frac{r}{\lambda}\left(\frac{r}{\lambda} - 1\right) \cdot \left(\frac{r}{\lambda} - 2\right)}{2 . 3}(A\lambda + B\lambda^2 + C\lambda^3 + \&c.)^3 + \ldots + \&c.$$

$$= 1 + r(A + B\lambda + C\lambda^2 + \&c.) +$$

$$\frac{r(r - \lambda)}{2}(A + B\lambda + C\lambda^2 + \&c.)^2 +$$

$$\frac{r(r - \lambda) . (r - 2\lambda)}{2 . 3}(A + B\lambda + C\lambda^2 + \&c.)^3 + \ldots + \&c.$$

If now we assume $\lambda = 0$, this expression becomes

$$b^r = 1 + Ar + \frac{A^2 r^2}{2} + \frac{A^3 r^3}{2 . 3} + \ldots + \&c.$$

Which expression, being perfectly general, is true whatever value is assigned to r; we may therefore substitute λ for r, whence we obtain

$$b^\lambda = 1 + A\lambda + \frac{A^2}{2}\lambda^2 + \frac{A^3}{2 . 3}\lambda^3 + \ldots + \&c. \ldots\ldots\ldots\ [2.]$$

The value of A is already known in terms of y, but as $b =$

$1 + y$, therefore $y = b - 1$, and if we substitute this value for y, we have

$$A = \left\{ (b-1) - \frac{1}{2}(b-1)^2 + \right.$$

$$\left. \frac{1}{3}(b-1)^3 - \frac{1}{4}(b-1)^4 + \dots\dots\dots\dots - \&c. \right\} \dots\dots [3.]$$

Proposition C.

Problem. *From the equation*

$$b^{\lambda} = n = 1 + A\lambda + \frac{A^2}{2}\lambda^2 + \frac{A^3}{2.3}\lambda^3 + \dots\dots + \&c.$$

to determine the value of λ, *in terms of* b *and* n.

If, in the equation $n = b^{\lambda}$, both sides are raised to the power of x, it becomes $n^x = b^{\lambda x}$; then expanding n^x in terms of x, we obtain (Prop. B, [2])

$$n^x = 1 + A_1 x + \frac{A_1^2}{2} x^2 + \frac{A_1^3}{2.3} x^3 + \dots\dots\dots + \&c.$$

in which (Prop. B, [3])

$$A_1 = \left\{ (n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \dots\dots + \&c. \right\}$$

Also expanding $b^{\lambda x}$ in terms of λx, we have (Prop. B, [2])

$$b^{\lambda x} = 1 + A\lambda x + \frac{A^2}{2}\lambda^2 x^2 + \frac{A^3}{2.3}\lambda^3 x^3 + \dots\dots + \&c.$$

If now in the equation $b^{\lambda x} = n^x$, we substitute the values of $b^{\lambda x}$ and n^x obtained above, it becomes

$$1 + A\lambda x + \frac{1}{2}A^2\lambda^2 x^2 + \frac{1}{6}A^3\lambda^3 x^3 + \dots + \&c.$$

$$= 1 + A_1 x + \frac{1}{2}A_1^2 x^2 + \frac{1}{6}A_1^3 x^3 + \dots + \&c.$$

From which we have, by Prop. A,

$$A\lambda = A_1$$
$$A^2\lambda^2 = A_1^2$$
$$A^3\lambda^3 = A_1^3, \ \&c.$$

From each of which we obtain

$$\lambda = \frac{A_1}{A}.$$

Substituting for A_1 its value given above, and for A its value determined in Prop. B, [3], we have

$$\lambda = \frac{(n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \&c.}{(b-1) - \frac{1}{2}(b-1)^2 + \frac{1}{3}(b-1)^3 - \&c.} \ldots\ldots [1.]$$

SCHOLIUM. Since $b^\lambda = n$, it follows, from the definition of a logarithm, page 4, that λ is the logarithm of the number n to the base b. Now as b may have any value that we please assigned to it, and, as every different value of b gives a different value of λ, it follows that there may be any number of logarithms corresponding with the number n, because any number of values may be given to the base b*.

We may therefore assume such a value for b as shall give

$$A = \left\{ (b-1) - \frac{1}{2}(b-1)^2 + \frac{1}{3}(b-1)^3 - \&c. \right\} = 1,$$ in

which case the expression [1, above] for the logarithm becomes

$$\lambda = (n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \ldots + \&c. \ldots. [2.]$$

This is the value for A actually taken by Baron Napier, and employed by him in his first Table of Logarithms, from which circumstance, logarithms calculated to this base are termed *Napierean Logarithms.*

PROPOSITION D.

PROBLEM. *In the equation*

$$b^\lambda = 1 + A\lambda + \frac{A^2}{2}\lambda^2 + \frac{A^3}{2.3}\lambda^3 + \frac{A^4}{2.3.4}\lambda^4 + \ldots + \&c.$$

to determine the value of b, when A is made equal to unity.

Substituting the assumed value of A, in the above expression, it becomes

* See page 8.

$$b^{\lambda} = 1 + \lambda + \frac{1}{2}\lambda^2 + \frac{1}{2.3}\lambda^3 + \frac{1}{2.3.4}\lambda^4 + \ldots + \&c.$$

Now, as this expression is true, whatever be the value of λ, it is true when $\lambda = 1$, in which case it becomes

$$b = 1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \frac{1}{2.3.4.5} + \&c.$$

From which expression, as it is rapidly convergent, we may easily determine the value of b, to any required degree of exactness.

SCHOLIUM. The following is the calculation for the first thirteen terms.

1 + 1 =	2·000,000,000	000
½ =	·500,000,000	000
÷ by 3 =	·166,666,666	667
„ 4 =	·041,666,666	667
„ 5 =	·008,333,333	333
„ 6 =	·001,388,888	889
„ 7 =	·000,198,412	698
„ 8 =	·000,024,801	587
„ 9 =	·000,002,755	732
„ 10 =	·000,000,275	573
„ 11 =	000,000,025	052
„ 12 =	·000,000,002	088
	2·718,281,828	286

In which the first nine decimals are correct. This number is usually denoted by ϵ, and is, as stated in the scholium to the preceding proposition, the base of the Napierean system of logarithms.

PROPOSITION E.

PROBLEM. *To obtain a rapidly convergent series, for calculating the logarithms of numbers.*

In the expression Prop. C, [1], if we put A for its equivalent value in the denominator, we have

$$\lambda = \log_b n = \frac{1}{A}\left\{(n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \ldots + \&c.\right\} \ldots\ldots\ldots\ldots\ldots\ [1.]$$

This expression being true for all values of n, we may put $1 + m = n$, it then becomes

$$\log_b(1+m) = \frac{1}{A}\left(m - \frac{1}{2}m^2 + \frac{1}{3}m^3 - \dots + \&c.\right)$$

Again, if we put $1 - m = n$, we have

$$\log_b(1-m) = \frac{1}{A}\left(-m - \frac{1}{2}m^2 - \frac{1}{3}m^3 - \dots - \&c.\right)$$

Then, subtracting the second equation from the first, we obtain

$$\log_b(1+m) - \log_b(1-m) = \log_b\frac{1+m}{1-m} =$$

$$\frac{1}{A}\left(m - \frac{1}{2}m^2 + \frac{1}{3}m^3 - \dots + \&c.\right)$$

$$-\frac{1}{A}\left(-m - \frac{1}{2}m^2 - \frac{1}{3}m^3 - \dots - \&c.\right)$$

$$= \frac{2}{A}\left(m + \frac{1}{3}m^3 + \frac{1}{5}m^5 + \dots + \&c.\right)$$

Now, let $1 + m = a$, and $1 - m = a - 1$, then we have

$$\frac{1+m}{1-m} = \frac{a}{a-1}.$$

From which we obtain

$$m = \frac{1}{2a-1}.$$

Substituting this value of m in the above expression, we have

$$\log_b\frac{a}{a-1} = \frac{2}{A}\left\{\frac{1}{2a-1} + \frac{1}{3(2a-1)^3} + \frac{1}{5(2a-1)^5} + \dots + \&c.\right\} \dots\dots\dots\dots [2.]$$

a series which is rapidly converging.

Proposition F.

Problem. ***To reduce the logarithms of a system having one base, to those having a different base.***

We have, in Prop. E, [1],

$$\log_b n = \frac{1}{A}\left\{(n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \dots + \&c.\right\}$$

In which expression the coefficient $\frac{1}{A}$ is constant for every logarithm having b for its base, its value being entirely independent of n (Prop. B, [3]); and we have further shown (Prop. D) that, when the base of the system of logarithms is taken equal to $2\cdot718281828 = \varepsilon$, the value of A is reduced to unity; in this case, therefore, we have

$$\log_\varepsilon n = (n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \dots + \&c. \quad [1.]$$

This system, therefore, having ε for its base, has been called the *natural* system of logarithms, because it can be expressed in terms of n alone, and has unity for its coefficient.

Now, we have (Prob. B, [3])

$$A = (b-1) - \frac{1}{2}(b-1)^2 + \frac{1}{3}(b-1)^3 - \dots + \&c.,$$

an expression from which, by comparison with [1] above, we immediately perceive that $\log_\varepsilon b = A$, or that the value of A, the denominator of the constant coefficient $\frac{1}{A}$, for any system of logarithms to the base b, is equal to the Napierean logarithm of b. The constant coefficient $\frac{1}{A}$ is called the *modulus* of the system to which it belongs; and to reduce logarithms having one base to those having a different one, it is only necessary to divide them by the modulus of their own system, by which they become reduced to natural, or Napierean logarithms, and then to multiply them by the modulus of the system having the required base *.

* See page 9.

PROPOSITION G.

PROBLEM. *To find the Napierean logarithm of b, when b equals* 10.

In the expression (Prop. E, [2])

$$\log_b \frac{a}{a-1} = \frac{2}{A}\left\{\frac{1}{2a-1} + \frac{1}{3(2a-1)^3} + \frac{1}{5(2a-1)^5} + \dots + \&c.\right\}$$

if we put $a = 2$, and bear in mind that, as we want the Napierean logarithm, the modulus equals unity, we obtain

$$\log_e \frac{2}{1} = \log_e 2 = 2\left\{\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \dots + \&c.\right\}$$

Performing the calculation, we have

$$\begin{aligned}
\frac{1}{3} &= \cdot 333,333,333,333 \\
\frac{1}{3} \cdot \frac{1}{3^3} &= \cdot 012,345,679,012 \\
\frac{1}{5} \cdot \frac{1}{3^5} &= \cdot 000,823,045,268 \\
\frac{1}{7} \cdot \frac{1}{3^7} &= \cdot 000,065,321,053 \\
\frac{1}{9} \cdot \frac{1}{3^9} &= \cdot 000,005,645,029 \\
\frac{1}{11} \cdot \frac{1}{3^{11}} &= \cdot 000,000,513,184 \\
\frac{1}{13} \cdot \frac{1}{3^{13}} &= \cdot 000,000,048,248 \\
\frac{1}{15} \cdot \frac{1}{3^{15}} &= \cdot 000,000,004,646 \\
\frac{1}{17} \cdot \frac{1}{3^{17}} &= \cdot 000,000,000,436 \\
\hline
&\ \cdot 346,573,590,209 \\
&\ \qquad\qquad\qquad\quad 2 \\
\hline
\log_e 2 &= 0 \cdot 693,147,180,418.
\end{aligned}$$

Now, $4 = 2^2$, therefore $\log_e 4 = 2 \log_e 2 =$
$1{\cdot}386,294,360,836$.

Again, let us put $a = 5$, we have then

$$\log_e \frac{5}{4} = 2\left\{\frac{1}{9} + \frac{1}{3 \,.\, 9^3} + \frac{1}{5 \,.\, 9^5} + \frac{1}{7 \,.\, 9^7} + \ldots + \&c.\right\}$$

and, performing the calculation, we have

$$\begin{aligned}
\frac{1}{9} &= {\cdot}111,111,111,111 \\
\frac{1}{3} \cdot \frac{1}{9^3} &= {\cdot}000,457,247,371 \\
\frac{1}{5} \cdot \frac{1}{9^5} &= {\cdot}000,003,387,017 \\
\frac{1}{7} \cdot \frac{1}{9^7} &= {\cdot}000,000,029,869 \\
\frac{1}{9} \cdot \frac{1}{9^9} &= {\cdot}000,000,000,286 \\
\hline
& \ {\cdot}111,571,775,654 \\
& \phantom{= {\cdot}111,571,775,65}2 \\
\hline
\log_e \frac{5}{4} &= {\cdot}223,143,551,308
\end{aligned}$$

Now, the $\log_e \frac{5}{4} = \log_e 5 - \log_e 4$; if, therefore, we add to $\log_e \frac{5}{4}$ the $\log_e 4$, as found above, we shall have

$$\begin{array}{r}
{\cdot}223,143,551,308 \\
1{\cdot}386,294,360,836 \\
\hline
1{\cdot}609,437,912,144
\end{array}$$

equal the $\log_e 5$; then, since $\log_e 5 + \log_e 2 = \log_e 10$, we have

$$1{\cdot}609,437,912 + {\cdot}693,147,180 = 2{\cdot}302,585,092,$$

which is the true value of $\log_e 10$, to nine places of decimals.

Proposition H.

Theorem. *The logarithm of any number q to the base p, multiplied by the logarithm of p to the base q, is always equal to unity; that is,* $\log_p q \,.\, \log_q p = 1$.

Let $\log_p q = l$, and $\log_q p = k$; then $p^l = q$, and $q^k = p$.

If the first of these, $p^l = q$, be raised to the power of k, we have

$$p^{l \,.\, k} = q^k.$$

But $q^k = p$, therefore $p^{l \,.\, k} = p$, and

$$l \,.\, k = \log_p q \,.\, \log_q p, \text{ must equal unity.}$$

Proposition I.

Problem. *To determine the value of* $\log_b \varepsilon$, ε being the base of the Napierean system of logarithms.

Comparing the expression for the value of A, given in the Scholium to Proposition C, with the formula [2] in the same, we see that A is the Napierean logarithm of b, or $\varepsilon^{A} = b$; therefore $\log_\varepsilon b = A$.

Now we have, from Theorem H,

$$\log_\varepsilon b \,.\, \log_b \varepsilon = 1, \text{ or } A \,.\, \log_b \varepsilon = 1,$$

$$\text{Therefore } \log_b \varepsilon = \frac{1}{A} = \frac{1}{2.302585092} = \cdot 434294482.$$

This is therefore the value of the modulus [Prop. F] of the common system of logarithms.

Proposition K.

Theorem. *If a series of logarithms to the same base are in arithmetical progression, the corresponding numbers will form a series in geometrical progression.*

That is, if in $b^{l_1} = n_1$, $b^{l_2} = n_2$, $b^{l_3} = n_3$, $b^{l_4} = n_4$, the values of the exponents of b are such that l_1, l_2, l_3, l_4, form an *arithmetical* progression, then will n_1, n_2, n_3, n_4, form a *geometrical* progression.

For, let δ be the *common difference* of the arithmetical series, then

$$b^{l_2} = b^{l_1+\delta} = b^{l_1} . b^{\delta}$$

$$b^{l_3} = b^{l_2+\delta} = b^{l_2} . b^{\delta}$$

$$b^{l_4} = b^{l_3+\delta} = b^{l_3} . b^{\delta}$$

&c. &c. &c.

Let $b^{\delta} = n_{\delta}$, then, substituting in the above for b^{δ}, b^{l_1}, b^{l_2}, b^{l_3}, &c., their equals n_{δ}, n_1, n_2, n_3, &c., we obtain

$$n_2 = n_1 . n_{\delta}$$

$$n_3 = n_2 . n_{\delta}$$

$$n_4 = n_3 . n_{\delta}$$

&c. &c.

Or, we see that each term of the series n_1, n_2, n_3, &c., is equal to the preceding term *multiplied* by the constant quantity n_{δ}; they are, therefore (Def. 9), in *geometrical* progression, n_{δ} being their *common ratio.*

SCHOLIUM. It should be observed that, since $b^{\delta} = n_{\delta}$, δ, the *common difference* of the series of logarithms, is the logarithm of n_{δ}, the *common ratio* of the series of numbers.

PROPOSITION L.

PROBLEM. *To deduce an expression for the limit of the increment of a logarithm, produced by any given increase in the corresponding natural number.*

If, in the expression Prop. E [1], we put $\frac{1}{b} + 1$ for n, it becomes

$$\log\left(\frac{1}{b} + 1\right) = \frac{1}{A}\left\{\frac{1}{b} - \frac{1}{2\,b^2} + \frac{1}{3\,b^3} - \frac{1}{4\,b^4} + \dots \text{\&c.}\right\} \dots [1].$$

Now $\log\left(\frac{1}{b} + 1\right) = \log\left(\frac{1+b}{b}\right) = \log(b + 1) - \log b$,

equal the increment occasioned in the logarithm of b, by increasing its value by unity.

In the expression $\left\{\frac{1}{b} - \frac{1}{2\,b^2} + \frac{1}{3\,b^3} - \&c.\right\}$, the first term, $\frac{1}{b}$, is greater than the sum of all the succeeding terms, and therefore

$$\log\left(\frac{1}{b} + 1\right) = \log(b+1) - \log b < \frac{1}{A}\cdot\frac{1}{b} \;\ldots\; [2].$$

That is, the difference between the logarithms of two numbers differing by unity, is less than the modulus of the system divided by the lesser of those numbers.

SCHOLIUM 1. In the common system of logarithms, the modulus $= \frac{1}{A}$ has been shown [Proposition I] to be equal to $\cdot 434294482$; in this case, therefore, we have

$$\log(b+1) - \log b < \frac{\cdot 434294482}{b} \;\ldots\ldots\ldots\ldots\ldots\ldots\; [3].$$

SCHOLIUM 2. In the case of the logarithms of several consecutive numbers, each greater by unity than the preceding, putting m for the modulus of the system, we have

$$\log(b+1) - \log b < \frac{m}{b}$$

$$\log(b+2) - \log(b+1) < \frac{m}{b+1}$$

$$\log(b+3) - \log(b+2) < \frac{m}{b+2},$$

from which we see that, as the numbers increase, the *rate of increase* of their logarithms decrease; thus, the addition of unity to b increases its logarithm by $\frac{m}{b}$, while the addition of unity to $b+1$ increases its logarithm only $\frac{m}{b+1}$; when, however, b is a large number, b and $b+1$ are very nearly equal, and therefore the rate of increase of the logarithms may be considered as proportional to that of the correspond-

ing numbers, so long as the increment of the latter is small, as compared with the number itself.

Proposition M.

Theorem. *The sum of the logarithms of two numbers, is the logarithm of their product.*

Let $\lambda = \log_b m$, and $l = \log_b n$, then $b^\lambda = m$, and $b^l = n$

Now.

$$m \, . \, n = b^\lambda \, . \, b^l = b^{\lambda + l}.$$

And because

$$b^{\lambda + l} = m \, . \, n,$$

therefore $\lambda + l$ is the logarithm of $m \, . \, n$, to the base b; or, *the sum of the logarithms of m and n is the logarithm of their product.*

Proposition N.

Theorem. *The logarithm of the quotient of two numbers is equal to the logarithm of the dividend, with the logarithm of the divisor subtracted from it.*

Let λ and l denote the same as in the foregoing proposition. Then

$$\frac{m}{n} = \frac{b^\lambda}{b^l} = b^{\lambda - l}.$$

And because

$$b^{\lambda - l} = \frac{m}{n}$$

therefore $\lambda - l$ is the logarithm of $\frac{m}{n}$, to the base b; or, *the logarithm of the quotient of m divided by n, is equal to the logarithm of m, with the logarithm of n subtracted from it.*

Proposition O.

Theorem. *The logarithm of any power of a number, is equal to the logarithm of that number, multiplied by the exponent of the power.*

Let $\lambda = \log_b m$, then

$$m = b^{\lambda},$$
$$m^2 = b^{\lambda} \cdot b^{\lambda} = b^{2\lambda},$$
$$m^3 = b^{\lambda} \cdot b^{\lambda} \cdot b^{\lambda} = b^{3\lambda},$$
$$m^n = b_1^{\lambda} \cdot b_2^{\lambda} \cdot b_3^{\lambda} \cdot \ldots \cdot b_n^{\lambda} = b^{n\lambda}.$$

And because

$$b^{n\lambda} = m^n,$$

therefore $n\lambda$ is the logarithm of m^n to the base b; or, *the logarithm of the nth root of m, is equal to n times the logarithm of m.*

Proposition P.

Theorem. *The logarithm of any root of a number, is equal to the logarithm of that number, divided by the exponent of the root.*

Let $\lambda = \log_b m$, then $m = b^{\lambda}$; let the square root of $m = x$, and the logarithm of $x = l$, then

$$m = x \cdot x = b^{2l} = b^{\lambda},$$

$$\text{therefore, } 2l = \lambda, \text{ and } l = \frac{\lambda}{2}.$$

In like manner, if the cube root of $m = y$, and the logarithm of $y = p$, then

$$m = y \cdot y \cdot y = b^{3p} = b^{\lambda};$$

$$\text{therefore, } 3p = \lambda, \text{ and } p = \frac{\lambda}{3}.$$

And generally, if the nth root of $m = z$, and the logarithm of $z = q$, then

$$m = z_1 \cdot z_2 \cdot z_3 \cdot \ldots \ldots z_n = b^{nq} = b^{\lambda};$$

$$\text{therefore, } nq = \lambda, \text{ and } q = \frac{\lambda}{n};$$

or, *the logarithm of the nth root of m, is equal to the logarithm of m, divided by n.*

Proposition Q.

Theorem. *In the system of logarithms whose base is* 10, *the mantissa is the same for the same order of figures, whether those figures are integers or decimals.*

If the figures composing the two numbers are in the same order, and only differ in the place of the decimal point, the two numbers may be made equal by altering the position of the point in one of them, which will, in effect, be *multiplying* or dividing by 10, for every place that the decimal is moved to the *right* or to the left.

The logarithm of the number which has thus been *multitiplied* or divided by 10, or some integral power of 10, must have the logarithm of that power of 10 *added to* or subtracted from it, in order to be still the correct logarithm of that number; and it will then become equal to the logarithm of the other number.

Now, the logarithm of any power of 10 to the base 10, is obviously the *exponent* of that power, and as the exponent is integral, the logarithm of every integral power of 10 must itself be an integer, with no decimals or mantissa. The *addition* or subtraction, therefore, of the logarithm of the power of 10, by which the number has been *multiplied* or divided, will not affect its mantissa, which will consequently be the same as before its value was altered. And therefore the mantissæ of the two logarithms were originally the same.

Proposition R.

Theorem. *The characteristic of the logarithm of a number to the base* 10, *is always one less than the number of integral figures in that number.*

Let the number consist of only one integer; then its value must be less than 10; now the logarithm of 10 is 1·, therefore the logarithm of the number must be less than that of 10, and therefore its characteristic must be 0 (followed by some decimal), and in this case is one less than the number of integers in the number.

Then, let the number be successively multiplied by 10, and at the same time let the logarithm of 10, or 1, be successively added to the characteristic of its logarithm.

Now each multiplication by 10, will add an integer to the number, at the same time that it will add 1 to the characteristic of its logarithm, and therefore as originally the character-

istic was 1 less than the number of integers, so it will always continue, however great the number of integers may be.

PROPOSITION S.

THEOREM. *In the logarithm (to the base* 10) *of a number less than unity, the characteristic is negative, but the mantissa is positive; and the value of the characteristic is one greater than the number of cyphers between the decimal point and the first significant figure, the number being decimally expressed.*

Let the number be multiplied by such a power of 10, as will make it have only one integral figure, and let that power be the nth; then the logarithm will have been increased by n, and it will now have 0 for a characteristic (Prop. R), followed by a mantissa; both being positive. Let m equal the value of the mantissa, now this must have been also its original value (Prop. Q); therefore, since the logarithm now equals m, and n has been added to it, its original value must have been $m-n$; that is, it must have had a positive mantissa equal to m, and a negative characteristic equal to n.

Now the number of times (or n) that a decimal fraction must be multiplied by 10, to make only its first significant figure an integer, must be one greater than the number of cyphers which originally stood between that significant figure and the decimal point.

Therefore the negative characteristic of the logarithm of a decimal fraction, is one greater than the number of cyphers between the first significant figure and the decimal point.

CHAPTER V.

Description of Logarithmic Tables.

BEFORE proceeding to give rules for performing the various processes of Logarithmic Arithmetic, it will be advantageous to describe generally, a few of the most useful Tables of Logarithms, so as to render the student familiar with their use, before he is actually required to employ them.

The object of Mathematical Tables is to present in a concise form, and one easily referred to, two or more series of numbers mutually dependent upon each other. So that any number in one series being given, the corresponding number

in either of the other series may be immediately found, on inspection of the tables. The number given is termed the *argument* of the tables, and the number sought the *resultant*. Thus in the table a specimen of which is given at page 33, the numbers in the left-hand margin and at the head of the table are the *argument*, by which we are directed where to find the logarithms of those numbers, which logarithms are the *resultants*. When we thus seek in any column of a table for the argument by which to find some other number, we are said to *enter* that column with the argument. For example, if we are looking in the table at page 33 for the logarithm of 2565, we *enter* the column of numbers (distinguished by N. at the top) with the *argument* 2565, and on the same line in the contiguous column we find the *resultant* 4090874, which is the logarithm required.

Tables of the Logarithms of Numbers exist under a great variety of forms, and are calculated to a greater or less number of decimal places, according to the purposes to which they are intended to be applied. For Astronomical and Trigonometrical calculations, where considerable accuracy is required, tables are used in which the logarithms are carried to seven places of decimals; for ordinary purposes tables of six places will be found ample; and even in many cases five places will be sufficient. We shall describe some of the best and most generally employed tables, to seven, six, and five places.

The best tables of the Logarithms of Numbers to seven places, are those by Babbage, although for general use we should recommend Hutton's, which contain logarithmic sines, tangents, &c., to the same number of places, also the natural sines and tangents, and a great variety of other tables which will be found of frequent use. We have given on the opposite page as a specimen of these tables, a portion of one page of the same.

N. 25500 L. 406 OF NUMBERS. 37

N.	0	1	2	3	4	5	6	7	8	9	D.	Pro.
2550	4065402	5572	5742	5913	6083	6253	6424	6594	6764	6934		
51	7105	7275	7445	7615	7786	7956	8126	8296	8466	8637		170
52	8807	8977	9147	9317	9487	9658	9828	9998	0168	0338		1 17
53	4070508	0678	0848	1018	1189	1359	1529	1699	1869	2039		2 34; 3 51
54	2209	2379	2549	2719	2889	3059	3229	3399	3569	3739	170	4 68; 5 85
55	3909	4079	4249	4419	4589	4759	4929	5099	5269	5439		6 102; 7 119
56	5608	5778	5948	6118	6288	6458	6628	6798	6968	7137		8 136; 9 153
57	7307	7477	7647	7817	7987	8156	8326	8496	8666	8836		
58	9005	9175	9345	9515	9684	9854	0024	0194	0363	0533		
59	4080703	0873	1042	1212	1382	1551	1721	1891	2060	2230		
2560	2400	2569	2739	2909	3078	3248	3417	3587	3757	3926		
61	4096	4265	4435	4604	4774	4944	5113	5283	5452	5622		
62	5791	5961	6130	6300	6469	6639	6808	6978	7147	7317		
63	7486	7656	7825	7994	8164	8333	8503	8672	8841	9011		
64	9180	9350	9519	9688	9858	0027	0196	0366	0535	0704		169
65	4090874	1043	1212	1382	1551	1720	1889	2059	2228	2397		1 17; 2 34
66	2567	2736	2905	3074	3243	3413	3582	3751	3920	4089		3 51; 4 68
67	4259	4428	4597	4766	4935	5105	5274	5443	5612	5781		5 85; 6 101
68	5950	6119	6288	6458	6627	6796	6965	7134	7303	7472	169	7 118; 8 135
69	7641	7810	7979	8148	8317	8486	8655	8824	8993	9162		9 152

The natural numbers which form the *arguments* of the table extend from 10000 to 107999, the *resultants*, or logarithms answering to them, from 4·0000000 to 5·03341973, the former being given to 5 and 6 places, while the latter extend to 7 and 8 places. In the extreme left-hand column headed N, which is the column of arguments, only the first four figures of the natural numbers are given, the last figure must be sought for along the top of the table, in the line of figures immediately under the words "Logarithm of Numbers;" and the resultant, or the logarithm itself, will be found at the intersection of the two lines in which the two portions of the argument were found; that is, on the same *line* with the *first four* figures, and in the same *column* as the *last* figure. We observe, however, upon looking at the table, that while the first column of resultants (having 0 at the top) contains 7 figures, the other nine columns contain only 4. The explanation of this is as follows, the four figures given in these columns are only the four final figures of the logarithm, that is the 4th, 5th, 6th, and 7th, decimals, the first three figures, or the 1st, 2nd, and 3rd, decimals, are the same as those of the logarithms in the first column of resultants, and being the same it is considered unnecessary to repeat them, as they may be as easily supplied from the first column, and considerable saving of space is effected by their omission. It is in order to allow of this saving of space, by the omission of the similar figures, that the peculiar arrangement of the tables, by which a portion of the argument is found in the side column and a portion at the head of the table, has been adopted. It is not, however, always the case, that the initial figures found in the first column are the correct initial figures for all the other logarithms in the same line, because as the logarithms successively increase, after a certain interval, the last of the initial figures or the 3rd decimal becomes altered in value, and this alteration is equally likely to occur in any one of the columns. An example of this occurs at the third line of the table, in which the initial figures 406 apply, as far as the column headed with 7, they here, however, change, and in the next column become 407, and so continue until the ninth line, in which they change to 408 in the column headed with 6. Various methods have been adopted for directing attention to this change in the initial figures; in Hutton's tables it is shown by a line being drawn over the first figure of each of the logarithms to which the altered initial figures are to be applied, and in some other tables, as in Babbage's, it is

shown by the first cypher being put in smaller type, as | o168 | o338 |.

It has been shown, in the conclusion of the previous chapter (Scholium 2, Prop. L), that with small increments in the natural numbers, the logarithms corresponding with them increase in arithmetical progression, so that the difference between the successive logarithms remains constant for several logarithms in succession. Whenever the value of the difference changes, it is inserted in a column headed D, on the right of the table on the line in which the change occurs. Thus the number 170 is inserted in the column D, on the fifth line, and indicates that the difference between two successive logarithms has changed from 171 to 170 in the line in which it stands. The differences change much more rapidly at the commencement of the table than near its conclusion. The difference given in this column is that due to an increment of one unit in the 5th figure of the natural number, thus

$$\begin{array}{rrl} \text{Log of} & 25584 & = 4{\cdot}4079684 \\ ,, & \underline{25585} & = \underline{4{\cdot}4079854} \\ & \underline{\underline{1}} & \quad\;\; \underline{\underline{170}} \end{array}$$

and as for any increment less than this, we may consider the logarithms to vary in arithmetical progression, to ascertain the logarithm of any number between those given above, the increment of the logarithm to be added to 4·4079684 will bear the same proportion to 170, that the increment of the natural number does to 1; for example, let it be required to find the logarithm of 25584·6, here the increment of the number being ·6, we form the proportion 1·0 : ·6 : : 170 : 102, by which we find that 102 is the corresponding increment of the logarithm, which being added to 4·4079684 gives 4·4079786 for the logarithm of 25584·6. Again, if the increment of the natural number had been ·06, the corresponding increment of the logarithm would have been 10·2.

If now we divide the whole difference 170 by 10, we obtain 17, the difference corresponding with an increase of one unit in the sixth figure of the natural number, the double of this or 34 for two units, the treble or 51 for three units, and so on; and each of the numbers so obtained will be the increment of the logarithm corresponding with an increase of that number of units in the sixth figure of the natural number. The increment thus obtained, for each of the nine units, is

inserted in an adjoining column (headed "Pro.," an abbreviation of Proportional Parts).

The numbers contained in these little tables are, as already explained, the increments of the logarithm for an increase in the *sixth* figure of the natural numbers, they express, however, the increments for the units in the *seventh* place of the natural number when divided by 10, or for the *eighth* when divided by 100. Thus, suppose the logarithm of 25608587 were required, we derive at once from the table the logarithm of the first five figures, to which we add the proper increment for each additional figure, derived from the little table in the right-hand column. Thus—

Log of .	25608000	is	7·4083757
Increment for	500	,,	85
,,	80	,,	13·6
,,	7	,,	1·19
Therefore the log of	25608587	,,	7·4083857

These little tables of proportional parts are of equal service in finding the natural numbers corresponding with any given logarithm. Thus, if the logarithm given were 4·4074327, on looking in the table we see that the next less logarithm is 4·4074249, which corresponds with the natural number 25552; then subtracting the logarithm taken from the table, from the given logarithm, we obtain the difference, 78; looking in the second column of the table of Proportional Parts, we find against the next less difference, 68, the number 4, which is the sixth figure of the number required; we have still 10 left, to which adding a nought we obtain 100, and the nearest number in the table being 102, against which we find 6, that is the seventh figure required. The number answering to the logarithm 4·4074249 is therefore 25552·46.

In these and all the best tables of logarithms, the characteristic is omitted, the tables containing only the mantissa of the logarithm. The characteristic must be added in accordance with the rule given at page 11.

We next pass on to describe tables of logarithms to *six* decimal places. As a specimen, we have given a page from the "Mathematical Tables," forming one of the same series as the present work.

N.	0	1	2	3	4	5	6	7	8	9	N.
240	380211	0392	0573	0754	0934	1115	1296	1476	1656	1837	240
1	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	1
2	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	2
3	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	3
4	7390	7568	7746	7924	8101	8279	8456	8634	8811	8989	4
5	9166	9343	9520	9698	9875	0051	0228	0405	0582	0759	5
6	390935	1112	1288	1464	1641	1817	1993	2169	2345	2521	6
7	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	7
8	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	8
9	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	9
250	7940	8114	8287	8461	8634	8808	8981	9154	9328	9501	250
1	9674	9847	0020	0192	0365	0538	0711	0883	1056	1228	1
2	401401	1573	1745	1917	2089	2261	2433	2605	2777	2949	2
3	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	3
4	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	4
5	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070	5
6	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	6
7	9933	0102	0271	0440	0609	0777	0946	1114	1283	1451	7
8	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132	8
9	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	9
260	4973	5140	5307	5474	5641	5808	5974	6141	6308	6474	260
1	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	1
2	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791	2
3	9956	0121	0286	0451	0616	0781	0945	1110	1275	1439	3
4	421604	1768	1933	2097	2261	2426	2590	2754	2918	3082	4
5	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	5
6	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	6
7	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	7
8	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	8
9	9752	9914	0075	0236	0398	0559	0720	0881	1042	1203	9

PROPORTIONAL PARTS.

N.	1	2	3	4	5	6	7	8	9
2433	17·8	35·6	53·4	71·2	89·0	106·8	124·6	142·4	160·2
2446	17·7	35·4	53·1	70·8	88·5	106·2	123·9	141·6	159·3
2460	17·6	35·2	52·8	70·4	88·0	105·6	123·2	140·8	158·4
2474	17·5	35·0	52·5	70·0	87·5	105·0	122·5	140·0	157·5
2488	17·4	34·8	52·2	69·6	87·0	104·4	121·8	139·2	156·6
2503	17·3	34·6	51·9	69·2	86·5	103·8	121·1	138·4	155·7
2517	17·2	34·4	51·6	68·8	86·0	103·2	120·4	137·6	154·8
2532	17·1	34·2	51·3	68·4	85·5	102·6	119·7	136·8	153·9
2547	17·0	34·0	51·0	68·0	85·0	102·0	119·0	136·0	153·0
2562	16·9	33·8	50·7	67·6	84·5	101·4	118·3	135·2	152·1
2577	16·8	33·6	50·4	67·2	84·0	100·8	117·6	134·4	151·2
2592	16·7	33·4	50·1	66·8	83·5	100·2	116·9	133·6	150·3
2608	16·6	33·2	49·8	66·4	83·0	99·6	116·2	132·8	149·4
2624	16·5	33·0	49·5	66·0	82·5	99·0	115·5	132·0	148·5
2640	16·4	32·8	49·2	65·6	82·0	98·4	114·8	131·2	147·6
2656	16·3	32·6	48·9	65·2	81·5	97·8	114·1	130·4	146·7
2672	16·2	32·4	48·5	64·8	81·0	97·2	113·4	129·6	145·8
2689	16·1	32·2	48·3	64·4	80·5	96·6	112·7	128·8	144·9

This table contains the logarithms of every number less than 10,000 to six places of decimals, and in their general form and arrangement are very similar to those just described. The natural numbers which form the argument of the table are given to four places, the first three being found in the left-hand column, and the fourth at the head of the table; the first three figures are also repeated in the last column, to facilitate the use of the tables. In the first column of resultants the whole six figures of the logarithm are given, but in the succeeding columns only the last four, the two initial figures being supplied from the first column. In these tables, a horizontal line is introduced to separate the logarithms which have different initial figures. the line being made to break, or step up, when the change in the initial figures occurs other than at the commencement of a line. Thus in the middle of the sixth line the initial figures change from 38 to 39, and this is indicated by the line thus 9875 | 0051, the former of these being 389875, and the latter 390051.

In these tables also the proportional parts are somewhat differently arranged. In Hutton's and other logarithmic tables, the line in which the difference changes its value is shown, but each line contains ten logarithms, and there is nothing to indicate between which of these logarithms the change occurs; in the tables now being described, the number corresponding with the logarithm at which the change takes place is given in the left-hand column, and on the same line will be found the proportional parts for each unit constituting the fifth figure of the natural number. Thus, let the logarithm of 246057 be required; here we obtain the logarithm of the first four figures at once from the body of the table, for the increment to be added for the other two figures we look in the table of proportional parts, and on the same line with the first four figures of the given number, and in the same column as the fifth figure of the same, we find the proportional part to be added for that figure, and on the same line and in the same column as the sixth figure, we find the proportional part, *which, having first been divided by ten*, must be added for that figure. Thus—

The log of .	246000	is	5·390935
Increment for	50	,,	88
,,	7	,,	12·32
Therefore the log of	246057	,,	5·391035

If the first four figures of the number are not exactly found the first column of proportional parts, we must take the xt less number. Thus, had the given number, whose garithm was required, been 254371, we must have looked r the proportional part for the last two figures in the line aving 2532 in the left-hand column, those being the next ss numbers to 2543.

To find a number from its logarithm, the application of this table is very simple. We must take the next less logarithm in the upper part of the table, and the four first figures of the corresponding number will be obtained; we must then take the difference between the given logarithm and that found in the table, and, looking in the table of proportional parts, on the same line with the first four figures just found (or the next less to them), for the next less number to this difference, the figure at the head of the column in which it is found will be the *fifth* figure of the required number. Then, if the difference found in the table be taken from the difference sought for, and a nought be added, the number at the head of the column in which this second difference may be found (on the same line as before), will be the *sixth* figure of the required number. Thus, what is the natural number whose logarithm is 3·416369?

The given logarithm .	= 3·416369		
Next less logarithm .	= 3·416308	= the log of 2608·	
	61	= 1st diff.	
Next less diff. in table . .	= 49·8	=	·3
	112	= 2nd diff.	
Nearest diff. in table . . .	= 116	=	·07
		Therefore the number required is	2608·37

Again, what is the natural number whose logarithm is 5·394564?

The given logarithm	= 5·394564		
Next less logarithm .	= 5·394452	= log of 248000	
	112	= 1st diff.	
Next less diff. in table . .	= 105	=	60
	70	= 2nd diff. =	4
		Therefore the required number is	248064

The next table of logarithms which we shall describe are those reprinted, under the superintendence of the Society for the Diffusion of Useful Knowledge, from the tables of Lalande published in France. A specimen of these tables is given below.

3960″ = 1°6′ 0″			3990″ = 1° 6 30″			4020″ = 1° 7′ 0″		
Num.	Log.	D.	Num.	Log.	D.	Num.	Log.	D.
3960	·59770	10	3990	·60097	11	4020	·60423	10
3961	·59780	11	3991	·60108	11	4021	·60433	11
3962	·59791	11	3992	·60119	11	4022	·60444	11
3963	·59802	11	3993	·60130	11	4023	·60455	11
3964	·59813	11	3994	·60141	11	4024	·60466	11
3965	·59824	11	3995	·60152	11	4025	·60477	10
3966	·59835	11	3996	·60163	10	4026	·60487	11
3967	·59846	11	3997	·60173	11	4027	·60498	11
3968	·59857	11	3998	·60184	11	4028	·60509	11
3969	·59868	11	3999	·60195	11	4029	·60520	11
3970	·59879	11	4000	·60206	11	4030	·60531	10
3971	·59890	11	4001	·60217	11	4031	·60541	11
3972	·59901	11	4002	·60228	11	4032	·60552	11
3973	·59912	11	4003	·60239	10	4033	·60563	11
3974	·59923	11	4004	·60249	11	4034	·60574	10
3975	·59934	11	4005	·60260	11	4035	·60584	11
3976	·59945	11	4006	·60271	11	4036	·60595	11
3977	·59956	10	4007	·60282	11	4037	·60606	11
3978	·59966	11	4008	·60293	11	4038	·60617	10
3979	·59977	11	4009	·60304	10	4039	·60627	11
3980	·59988	11	4010	·60314	11	4040	·60638	11
3981	·59999	11	4011	·60325	11	4041	·60649	11
3982	·60010	11	4012	·60336	11	4042	·60660	10
3983	·60021	11	4013	·60347	11	4043	·60670	11
3984	·60032	11	4014	·60358	11	4044	·60681	11
3985	·60043	11	4015	·60369	10	4045	·60692	11
3986	·60054	11	4016	·60379	11	4046	·60703	10
3987	·60065	11	4017	·60390	11	4047	·60713	11
3988	·60076	10	4018	·60401	11	4048	·60724	11
3989	·60086	11	4019	·60412	11	4049	·60735	11
3990	·60097		4020	·60423		4050	·60746	

They are only carried to five decimal places, and their arrangement is quite different from that of the tables already described. They contain the logarithms of every consecutive number from 1 to 10,000, the arguments and resultants being placed in parallel columns, and the differences between the logarithms being given in a third column on the right hand. In these tables no proportional parts of the differences are given for the several units in the fifth place of the natural number, but they have to be found by proportion in the manner explained at page 35.

Thus, suppose the logarithm of 39694 were required: we immediately find, from the table, the logarithm of 39690 to be 4·59868, but we know this to be too small, and we want the proportional part of the whole difference, 11, to be added for the four units in the fifth place of the natural number. Now, the difference, 11, corresponds with an increase of ten units in the fifth figure of the number, therefore, as 10 : 11 :: 4 : 4·4, which is the proportional part required. The rule, therefore, for finding the proportional parts is as follows:—Multiply the difference given in the third column by all the figures of the natural number, except the first four, and point off as many decimals in the product as there were figures in the multiplier, the integral portion will be the proportional part to be added to the logarithm. In the example above we have $11 \times \cdot 4 = 4\cdot 4$, the integer of which being added to 4·59868, gives 4·59872 for the logarithm of 39694.

Again, what is the logarithm of 403567? The logarithm of 403500 is 5·60584, and the difference, 11, being multiplied by 67, is 737, from which pointing off *two* decimals, leaves the integer 7 to be added; therefore, the logarithm of 403567 is 5·60591.

To find a number answering to a logarithm, from these tables, proceed as follows:—Look for the next less logarithm, and the number answering to it will be the first four figures of the number required. Then take the difference between this logarithm and the one given; to this difference add as many cyphers as additional figures are required, and divide by the difference given in the third column of the table, the quotient will be the figures to be added to the first four already derived from the tables; the position of the decimal point will be determined by the value of the characteristic.

For example, what is the number answering to the logarithm 3·60428?

Logarithm given . .	= 3·60428		
Next less logarithm .	= 3·60423	=	4020
	50 ÷ 10	=	·5
			4020·5

Therefore 4020·5 is the number whose logarithm 3·60428.

Again, what is the number answering to the logarithm 4·60719?

Logarithm given . .	= 4·60719		
Next less logarithm .	= 4·60713	=	40470
	600 ÷ 11	=	5·4
			40475·4

Therefore 40475·4 is the number whose logarithm is 4·60719.

Having described some of the principal tables, and explained the method of using them, it will be desirable to show how many figures may be relied upon as accurate, in the results obtained by tables of five, six, and seven decimal places.

Let us have the logarithm 3·17284 given to five places of decimals: now the real value of this logarithm, if expressed to a greater number of places, might, for aught that can be known, be anything between 3·172835 and 3·172845, and might therefore differ from the logarithm given by very nearly ·000005; which then is the extreme limit of the difference which tables to five places will show; any difference less than this might occur without any change in the value of the logarithm, as given in the table.

It has been shown in Prop. L [3], page 27, that the difference between the logarithms of two numbers, which differ only by unity, is less than the modulus of the system divided by the lesser number, or, in the case of common logarithms, than ·434294482 divided by the lesser number. Now, the

difference between the true logarithm and that given to five places may, as we have shown above, be nearly equal to ·000005, which is therefore less than ·4342945 divided by the number, or the number is less than $\frac{\cdot 4342945}{\cdot 000005} = 86858\cdot 9$. That is to say, that unless the number, whose logarithm is given, is less than 86859, its value cannot be determined with certainty beyond *four* figures; but that if less than 86859, the first *five* figures derived from the table will be true.

In a similar way it may be shown that, when working with tables of logarithms to six decimal places, the first *six* figures of the result may be depended upon if less than 868589, but if greater, only the first *five* figures must be kept. And in the case of logarithms to seven decimal places, if the result is less than 8685890, *seven* places will be accurate, but if greater, only *six*. Generally, in any tables of logarithms, the result obtained may be considered accurate to as many figures as there are decimal places in the logarithms, provided the mantissa of the logarithm is less than ·9388, but if greater, then the result will only be accurate to one less number of figures than the decimals in the logarithm.

CHAPTER VI.

Logarithmic Arithmetic

We next proceed to the application of logarithms to the ordinary processes of arithmetic, and to illustrate and explain their general use for the purposes of calculation. The references following the rules show the proposition in Chapter IV., in which the rule is demonstrated.

To find the Arithmetical Complement of a Logarithm.

By the *arithmetical complement* of a logarithm is meant the remainder left by the subtraction of the logarithm from 10. Thus the arithmetical complement of 3·241735 is $10\cdot 000000 - 3\cdot 241735 = 6\cdot 758265$. Its great use is in division, as will be presently shown; for if, instead of subtracting a logarithm, we add its complement, and subtract 10. we obtain the same result. To find the arithmetical complement employ the following rule.

Rule.—*Subtract the first right-hand significant figure from* 10, *and all the others (including the characteristic when positive) from* 9; *when the characteristic is negative, it must be added to* 9.

Examples.

The arithmetical complement of	5·631642	is	4·368358
„ „	2·170630	„	7·829370
„ „	$\bar{1}$·217034	„	10·782966
„ „	$\bar{3}$·173680	„	12·826320
„ „	3·607218	„	6·392782
„ „	0·714000	„	9·286000

Multiplication.

Rule.—*To multiply two or more numbers together, add their logarithms, the sum will be the logarithm of their product* (Prop. M).

Examples.

Multiply 5631 by 42.

Logarithm of 5631 = 3·750586
„ 42 = 1·623249

5·373835
5·373831 = log of 236500

40 = 2

Answer = 236502

Multiply 52, 734, and 6 together.

Logarithm of 52 = 1·716003
„ 734 = 2·865696
„ 6 = 0·778151

5·359850
5·359835 = log of 229000

150 = 8

Answer = 229008

Multiply 61, 22, and 65 together.

Logarithm of 61 = 1·785330
„ 22 = 1·342423
„ 65 = 1·812913

4·940666 = log of 87230.

Division.

Rule.—*To divide one number by another, subtract the logarithm of the divisor from the logarithm of the dividend, and the remainder will be the logarithm of the quotient* (Prop. N).

Examples.

Divide 1164 by 4.

Logarithm of	1164 =	3·065953
„	4 =	0·602060
		2·463893 = log of 291.

Divide 116908 by 5314.

Logarithm of	116900 =	5·067815
Prop. part for	8 =	29·68
Logarithm of	116908 =	5.067845
„	5314 =	3·725422
		1·342423 = log of 22.

Instead of subtracting the logarithm of the divisor we may *add* its arithmetical complement, the result, with 10 subtracted from the characteristic, will as before be the logarithm of the quotient. Thus, in the example above, the arithmetical complement of 3·725422, the logarithm of the divisor, is 6·274578, which added to 5·067845, gives 1·342423, the same answer as before. This method will be found very convenient where it is desired to divide one number by several others; we have, in such a case, only to add to the logarithm of the dividend, the arithmetical complement of the logarithms of the several divisors, and subtract from the characteristic as many tens as there were divisors, the result will be the logarithm of the quotient.

Divide 579416 by 4, 23, and 47.

Logarithm of . . .	579400 =	5·762978
Proportional part for	10 =	7·5
„ „	6 =	4·5
Logarithm of . . .	579416 =	5·762990
Arith. comp. of log of	4 =	9·397940
„ „	23 =	8·638272
„ „	47 =	8·327902
		2·127104 = log of 134.

Proportion, or the Rule of Three.

Questions in proportion, or the rule of three, may be resolved with great facility with the aid of logarithms.

Rule.—*Add together the logarithms of the two middle terms, and from their sum subtract the logarithm of the first term, the remainder will be the logarithm of the fourth term, or quantity required. Or, instead of subtracting the logarithm of the first term*, add its arithmetical complement *and subtract* 10 *from the characteristic.*

Examples.

If 14 men, in 47 days, excavate 5631 cubic yards, what length of time will it take them to excavate 47280 cubic yards?

Or, as 5631 : 47 :: 47280 : ?

Logarithm of . . .	47280 =	4·674677
„ . . .	47 =	1·672098
		6·346775
Logarithm of . . .	5631 =	3·750586
		2·596189 = log of 394·626

By the second method:—

Logarithm of . . .	47280 =	4·674677
„ . . .	47 =	1·672098
Arith. comp. of log of	5631 =	6·249414
		2·596189 as before.

If an engine of 67 horses' power can raise from a reservoir 57,600 cubic feet of water in a given time, what horses' power will be required to raise 8,575,000 cubic feet in the same time?

Or, as 57,600 : 67 :: 8,575,000 : ?

Logarithm of . . .	8,575,000 =	6·933234
„ . . .	67 =	1·826075
		8·759309
Logarithm of . . .	57,600 =	4·760422
		3·998887 = log of 9974·4

Or:—

Logarithm of . . .	8,575,000 =	6·933234
„ . . .	67 =	1·826075
Arith. comp. of log of	57,600 =	5·239578
		3·998887 as before.

INVOLUTION AND EVOLUTION.

Involution is the process of raising a number to any power of itself, and evolution is the extraction of any root of a number; both these processes are very readily performed by means of logarithms.

RULE I.—*To raise a number to any power desired, multiply the logarithm of the number by the exponent of that power, and the product will be the logarithm of the power required.*

RULE II.—*To extract any root of a number, divide the logarithm of that number by the exponent of the root, and the quotient will be the logarithm of the root required.*

EXAMPLES.

What is the square of 745, the cube of 67, and the 7th power of 8?

$$\begin{array}{lr} \text{Logarithm of } 745 = & 2\cdot872156 \\ & 2 \\ \hline & 5\cdot744312 = \text{log of } 555025. \end{array}$$

$$\begin{array}{lr} \text{Logarithm of } 67 = & 1\cdot826075 \\ & 3 \\ \hline & 5\cdot478225 = \text{log of } 300763. \end{array}$$

$$\begin{array}{lr} \text{Logarithm of } 8 = & 0\cdot903090 \\ & 7 \\ \hline & 6\cdot321630 = \text{log of } 2097152 \end{array}$$

What is the square root of 4225, the cube root of 6859, and the 6th of 117649?

$$\begin{array}{l} \text{Log of } 4225 = 3\cdot625827 \div 2 = 1\cdot812913 = \text{log of } 65. \\ \text{Log of } 6859 = 3\cdot836261 \div 3 = 1\cdot278754 = \text{log of } 19. \\ \text{Log of } 117649 = 5\cdot070588 \div 6 = 0\cdot845098 = \text{log of } 7. \end{array}$$

When the number to be involved to any power, or whose root is to be extracted, is a *fraction*, its characteristic will be *negative*; in this case, in multiplying the logarithm by any number, it must be borne in mind that the mantissa is positive, and therefore that any figures carried from the multiplication of the same, must be *deducted* from the characteristic, instead of being added to it.

EXAMPLE.

What is the square of ·25, the cube of ·058, and the 5th power of ·9784?

$$\begin{array}{rl} \text{Logarithm of } \cdot 25 = & \bar{1}\cdot 397940 \\ & 2 \\ \hline & \bar{2}\cdot 795880 = \text{log of } \cdot 0625. \end{array}$$

$$\begin{array}{rl} \text{Logarithm of } \cdot 058 = & \bar{2}\cdot 763428 \\ & 3 \\ \hline & \bar{4}\cdot 290284 = \text{log of } \cdot 000195112. \end{array}$$

$$\begin{array}{rl} \text{Logarithm of } \cdot 9784 = & \bar{1}\cdot 990516 \\ & 5 \\ \hline & \bar{1}\cdot 952580 = \text{log of } \cdot 89656. \end{array}$$

In dividing a logarithm with a *negative* characteristic by any number, if the characteristic is a multiple of that number, or is divisible by it, proceed to divide in the usual manner, remembering, however, that the new characteristic will be *negative*. Should the characteristic not be divisible by the number by which it is required to divide the logarithm, separate the mantissa from the characteristic, and add to each such a number as will make the characteristic divisible, then divide each of the sums by the number, and the quotient will be the characteristic and mantissa respectively of the logarithm required. The equal numbers added to the character istic and mantissa, must of course be considered *negative* in the first case and positive in the other.

EXAMPLES.

What is the square root of ·209

Logarithm of ·209 = $\bar{1}\cdot 320146$

Adding $\dot{-}$ 1 to the characteristic we have $\bar{2} \div 2 = \bar{1}$, the new characteristic, and adding 1 to the mantissa we have $1\cdot 320146 \div 2 = \cdot 660073$ for the new mantissa, therefore $\bar{1}\cdot 320146 \div 2 = \bar{1}\cdot 660073$ = log of ·45716 = ne square root of ·209.

What is the cube root of ·000195112?

Logarithm of ·000195112 = $\bar{4}\cdot 290284$.

$\bar{4} + \bar{2} = \bar{6} \div 3 = \bar{2}$ for the characteristic, and $\cdot 290284 + 2 = 2\cdot 290284 \div 3 = \cdot 763428$ for the mantissa. Therefore, $\bar{4}\cdot 290284 \div 3 = \bar{2}\cdot 763428$ = log of ·058.

The four operations just described, namely, Multiplication, Division, Involution, and Evolution, comprise actually the

whole of the processes in the performance of which logarithms are employed, and when the student is conversant with these, he will be able to apply logarithms in a variety of cases in which their use will be attended with the saving of immense labour.

As an exercise in the preceding rules, and more especially in their practical application, we shall give a variety of useful formulæ, logarithmically expressed, and illustrate their use by examples; at the same time, that they may not be merely exercises, but may prove useful for reference, we shall arrange and classify them under their proper heads. In the following formulæ the letter λ will be used to denote "logarithm of;" thus, $\lambda\ a$, will mean the logarithm of a, or the quantity for which a stands; and $2\ \lambda\ (x^2 + y)$ means twice the logarithm of the quantity inclosed within the parenthesis, or y added to the square of x. All the lineal dimensions are given in feet, all the superficial dimensions in square feet, all the solid dimensions in cube feet, and all the weights or pressures in avoirdupois pounds, unless where it is otherwise expressly stated.

Interest.

Simple Interest.—[1.] Add together the logarithms of the principal, the rate, and the time, and from the sum subtract 2; the remainder will be the logarithm of the interest.

Compound Interest.—[2.] Find the amount of £1 at the given rate of interest for the first term; this is called the *ratio*, and the logarithm of the ratio for such rates of interest as are likely to be used are given in the annexed table. Multiply the logarithm of the ratio by the time, and add to the product the logarithm of the principal; the sum is the logarithm of the amount.

Rate of interest.	Logarithm of ratio.	Rate of interest.	Logarithm of ratio.
1	·0043214	5½	·0232525
1¼	·0053950	5¾	·0242804
1½	·0064660	6	·0253059
1¾	·0075344	6¼	·0263289
2	·0086002	6½	·0273496
2¼	·0096633	6¾	·0283679
2½	·0107239	7	·0293838
2¾	·0117818	7¼	·0303973
3	·0128372	7½	·0314085
3¼	·0138901	7¾	·0324173
3½	·0149403	8	·0334238
3¾	·0159881	8¼	·0344279
4	·0170333	8½	·0354297
4¼	·0180761	8¾	·0364293
4½	·0191163	9	·0374265
4¾	·0201540	9¼	·0384214
5	·0211893	9½	·0394141
5¼	·0222221	9¾	·0404045

Mensuration.

Triangle.—Let a, b, and c be the three sides, $d = \frac{1}{2}(a + b + c)$, and A equal the area; then

$$[3.]\quad \lambda\,A = \tfrac{1}{2}\left\{\lambda\,d + \lambda\,(d - a) + \lambda\,(d - b) + \lambda\,(d - c)\right\}.$$

Square.—[4.] The logarithm of the area equals twice the logarithm of one of the sides.

Rectangle.—[5.] The logarithm of the area equals the logarithm of the length added to the logarithm of the height.

Polygon.—Let l equal the length of one of the sides, n equal the number of sides, and A equal the area; then

$$[6.]^* \quad \lambda A = \cdot 39794 + 2\lambda l + \lambda n + \lambda \tan\left(\frac{90n - 180}{n}\right) - 1.$$

Circle.—Let d equal the diameter, c equal the circumference, and a equal the area; then

$$[7.] \quad \lambda d = \cdot 50285 + \lambda c - 1 = \cdot 60206 + \lambda a - \lambda c = \cdot 053455 + \tfrac{1}{2}\lambda a.$$

$$[8.] \quad \lambda c = \cdot 49715 + \lambda d = \cdot 60206 + \lambda a - \lambda d = \cdot 550605 + \tfrac{1}{2}\lambda a.$$

$$[9.] \quad \lambda a = \cdot 89509 + 2\lambda d - 1 = \cdot 90079 + 2\lambda c - 2 = \cdot 39794 + \lambda d + \lambda c - 1.$$

Circular arcs.—Let r equal the radius, m equal the measure of the arc in degrees, and l its length; then

$$[10.] \quad \lambda l = \cdot 2418776 + \lambda r + \lambda m - 2.$$

Circular sectors.—Let d equal the diameter, and a equal the area, the other letters as in [10]; then

$$[11.] \quad \lambda a = \cdot 69897 + \lambda r + \lambda l - 1 = \cdot 338456 + 2\lambda d + \lambda m - 3.$$

Parabola.—Let x_1 and x_2 be two abscissæ, y_1 and y_2 the corresponding ordinates, and a equal the area; then

$$[12.] \quad \lambda a = \cdot 823909 + \lambda x_2 + \lambda (2y_2) - 1.$$

$$[13.] \quad \lambda y_2 = \tfrac{1}{2}(\lambda x_2 + 2\lambda y_1 - \lambda x_1).$$

Ellipse.—Let t equal the transverse, and c the conjugate diameters, y equal any ordinate, and x_1, x_2, the corresponding abscissæ; also let a equal the area, and p equal the periphery; then

$$[14.] \quad \lambda a = \cdot 89509 + \lambda c + \lambda t - 1.$$

$$[15.] \quad \lambda p = \cdot 196118 + \lambda (t + c).$$

$$[16.] \quad \lambda y = \lambda c + \tfrac{1}{2}\lambda x_1 + \tfrac{1}{2}\lambda x_2 - \lambda t.$$

Formulæ [16] applies also in the case of the Hyperbola.

Parallelopipedon, prism, or *cylinder.*—[17.] The logarithm of the cubic contents equals the logarithm of the area of the base added to the logarithm of its perpendicular height.

Pyramid or Cone.—Let a equal the area of the base, h its perpendicular height, and s its solidity; then

$$[18.] \quad \lambda s = \cdot 823909 + \lambda a + \lambda h - 1.$$

Sphere.—Let d equal the diameter, c equal the circumference, s equal the solidity, and σ the surface; then

$$[19.] \quad \lambda \sigma = \lambda d + \lambda c = \cdot 696487 + 2\lambda d = \cdot 502837 + 2\lambda c - 1.$$

$$[20.] \quad \lambda s = \cdot 719 + 3\lambda d - 1 = \cdot 227372 + 3\lambda c - 1.$$

* The logarithmic tan must here be taken to a radius equal unity, therefore 10 must be subtracted from the characteristic given in the tables.

Regular Bodies.—Let l equal the length of any linear edge, s equal the solidity, σ equal the surface, and a and b, numbers obtained from the annexed table; then

$$[21.]\quad \lambda\sigma = 2\lambda l + a.$$

$$[22.]\quad \lambda s = 3\lambda l + b.$$

No. of sides.	Name.	a	b
4	Tetraëdron . . .	0·2385607	1·0713486
6	Hexaëdron . . .	0·7781513	0·0000000
8	Octaëdron . . .	0·5395906	1·6730624
12	Dodecaëdron . .	1·3148301	0·8844056
20	Icosaëdron . . .	0·9375306	0·3387940

Trigonometry.

Plane Triangles.—[23.] *Given two sides of a triangle and an angle opposite to one of them, to find the angle opposite to the other one.* Rule:—To the logarithmic sine of the given angle add the arithmetical complement of the logarithm of the opposite side, and the logarithm of the other given side; the sum with 10 subtracted from it will be the logarithmic sine of the angle required.

[24.] *Given two angles and a side opposite to one of them, to find the side opposite to the other one.* Rule:—To the logarithm of the given side, add the arithmetical complement of the logarithmic sine of its opposite angle, and the logarithmic sine of the other angle; the sum with 10 subtracted will be the logarithm of the side required.

[25.] *When two sides and the included angle are given, to find the third side.* Rule:—To the logarithm of the difference of the given sides add the arithmetical complement of the logarithm of their sum, and the logarithmic tangent of half the sum of the angles opposite the given sides, and the sum with 10 subtracted will be the logarithmic tangent of half the difference of those angles. Then to the arithmetical complement of the logarithmic cosine of half the said difference, add the logarithmic cosine of half the sum of the same angles, and the logarithm of the sum of the given sides; the sum with 10 subtracted will be the logarithm of the third side required.

[26.] *When the three sides are given, to find the angles.* Rule:—To the arithmetical complement of the logarithm of the longest side, add the logarithm of the sum of the other two sides, and the logarithm of the difference of those sides; the sum with 10 subtracted from it is the logarithm of the difference of the segments of the base or longest side. Then half this difference added to half the base will equal the longer segment, and deducted from it will equal the shorter one.

Right-angled triangles.—Let h equal the hypotenuse, b equal the base, and p equal the perpendicular; then

$$[27.]\ \lambda h = \tfrac{1}{2}\lambda(b^2 + p^2).$$

$$[28.]\ \lambda b = \tfrac{1}{2}\lambda(h^2 - p^2) = \tfrac{1}{2}\lambda(h + p) + \tfrac{1}{2}\lambda(h - p).$$

$$[29.]\ \lambda p = \tfrac{1}{2}\lambda(h^2 - b^2) = \tfrac{1}{2}\lambda(h + b) + \tfrac{1}{2}\lambda(h - b).$$

Mechanics.

Vis vivâ.—Let w equal the weight of a body, v its velocity in feet per second, and V its vis vivâ; then

$$[30.]\ \lambda V = 1{\cdot}507732 + \lambda w + 2\lambda v.$$

Action of gravity.—Let s equal the space passed over in t seconds, and v the velocity as above; then

$$[31.]\ \lambda s = {\cdot}69797 = \lambda t + \lambda v - 1 = 2{\cdot}205702 + 2\lambda t - 1$$
$$= {\cdot}190238 + 2\lambda v - 2.$$

$$[32.]\ \lambda v = 1{\cdot}507732 + \lambda t = {\cdot}30103 + \lambda s - \lambda t = {\cdot}904881 + \tfrac{1}{2}\lambda s.$$

$$[33.]\ \lambda t = {\cdot}492268 + \lambda v - 2 = {\cdot}30203 + \lambda s - \lambda v$$
$$= {\cdot}306649 + \tfrac{1}{2}\lambda s - 1.$$

Pendulums.—Let t equal the time in seconds of one vibration in a very small circular arc, and l the length; then

$$[34.]\ \lambda t = {\cdot}251016 + \tfrac{1}{2}\lambda l.$$

Central forces.—Let W equal the weight of a body moving in a circle whose radius is r, with a velocity of v feet per second, and let f equal the centrifugal force; then

$$[35.]\ \lambda f = {\cdot}492268 + 2\lambda v + \lambda W - \lambda r - 2.$$

Arches.—Let R equal radius of curvature at crown, b equal breadth of arch, w equal vertical weight on every square foot of the key-stone, including its own weight, and P equal the thrust or horizontal pressure on the key-stone; then

$$[36.]\ \lambda P = \lambda R + \lambda b + \lambda w.$$

Also let d equal horizontal distance of center of gravity of half the arch from its springing, r equal the rise of the arch, and W equal the weight of half the arch; then

$$[37.]\ \lambda P = \lambda W + \lambda d - \lambda r.$$

Retaining walls.—Let h equal height of wall, P equal pressure against wall, acting horizontally at one-third of the height of the wall above its base, and b a number obtained from the annexed table; then

$$[38.]\ \lambda P = 2\lambda h + b.$$

Material supported by wall.	b
Water	1·494850
Fine dry sand	1·194958
Loose shingle, perfectly dry	1·111867
Common earth, perfectly dry and pulverulent . . .	·945222
The same, slightly moistened, or in its natural state .	·747800
Earth, the most dense and compact.	·793301

Resistance of air.—Let a equal the area of a thin surface moving through water with a velocity equal v feet per second, and R equal the resistance; then

$$[39.]\quad \lambda R = \cdot 230449 + 2\lambda v + \lambda a - 2.$$

Resistance of water.—The notation being the same; then

$$[40.]\quad \lambda R = \cdot 98945 + 2\lambda v + \lambda a - 1.$$

HYDRAULICS.

Discharge through pipes.—Let d equal diameter in inches, Q equal quantity of water discharged in cubic feet per minute, l equal the length of the pipe, and h equal the head; then

$$[41.]\quad \lambda d = \tfrac{1}{5}\{2\lambda Q + \cdot 6515 + \lambda(l + 4\cdot 2\,d) - \lambda h - 2\}.$$

$$[42.]\quad \lambda Q = \tfrac{1}{2}\{1\cdot 3485 + \lambda h + 5\lambda d - \lambda(l + 4\cdot 2\,d)\}.$$

$$[43.]\quad \lambda l = 1\cdot 3485 + \lambda h + 5\lambda d - 2\lambda Q.$$

$$[44.]\quad \lambda h = \cdot 6515 + 2\lambda Q + \lambda(l + 4\cdot 2\,d) - 5\lambda d - 2.$$

Discharge through canals.—Let a equal sectional area of canal, p equal the wetted perimeter, l equal length, h equal corresponding fall, and v equal the velocity in feet per second; then

$$[45.]\quad \lambda v = 1\cdot 961142 + \tfrac{1}{2}\{\lambda a + \lambda h - \lambda p - \lambda l\}.$$

Discharge over weirs.—Let d equal the depth of water flowing over the weir, b equal its breadth, and Q equal the cubic feet discharged in a second; then

$$[46.]\quad \lambda Q = \cdot 511883 + \lambda b + \tfrac{3}{2}\lambda d.$$

STRENGTH OF MATERIALS.

Tensile strength.—Let a equal area in square inches, W equal weight producing fracture, and A equal number in column 2 of annexed table; then

$$[47.]\quad \lambda W = \lambda a + A.$$

Strength to resist Crushing.—Let a equal the area in square inches, W the weight producing fracture, and B numbers in column 3 of annexed table; then when the height of piece is between once and $4\tfrac{1}{2}$ times its diameter,

$$[48.]\quad \lambda W = \lambda a + B.$$

*Strength of Columns**.—Let W equal the breaking weight in tons, D equal external, and d internal diameter, both in inches, l equal the length, and C equal number in column 4 of annexed table; then when the column is *solid*, with both ends *rounded*, and its length not less than 15 times its diameter,

$$[49.]\quad \lambda W = 3\cdot 6\lambda D - 1\cdot 7\lambda l + C.$$

* Professor Hodgkinson's Formulæ.

When the column is *hollow;* then

$$[50.]\ \lambda\,W = \lambda\,(D^{3.6} - d^{3.6}) - 1.7\,\lambda\,l - .059243 + C.$$

When the column is *solid* both ends are *flat,* and the length is not less than 30 times the diameter; then

$$[51.]\ \lambda\,W = 3.6\,\lambda\,D - 1.7\,\lambda\,l + .471843 + C.$$

When the column is *hollow;* then

$$[52.]\ \lambda\,W = \lambda\,(D^{3.6} - d^{3.6}) - 1.7\,\lambda\,l + .473217 + C.$$

Transverse strength of a rectangular bar.—Let b equal the breadth and d the depth, both in inches, l equal the length, W the breaking weight, and D the number in the fifth column of the annexed table; then

$$[53.]\ \lambda\,W = \lambda\,b + 2\,\lambda\,d - \lambda\,l + D.$$

Transverse strength of Professor Hodgkinson's girder.—Let a equal area of bottom flange in inches, and d, W, and l have the same meaning as above; then

$$[54.]\ \lambda\,W = 3.685921 + \lambda\,a + \lambda\,d - \lambda\,l.$$

Deflexion.—Let δ equal the deflexion in inches with the weight W, and E equal the numbers in the sixth column of the annexed table; then

$$[55.]\ \lambda\,\delta = 3\,\lambda\,l + \lambda\,W - \lambda\,b - 3\,\lambda\,d - E.$$

Material.	A	B	C	D	E
Cast iron . . .	4·253338	5·032417	1·173186	3·310693	4·629338
Wrought iron . .	4·770499	......	1·414973	3·359836	4·761063
Steel	5·113943	......	1·574031	......	4·826910
Elm	3·988559	3·108565	......	2·528917	3·209515
Oak	4·074816	3·586587	0·209515	2·745855	3·527501
Fir	3·977724	......	0·068186	2·567026	3·428723

The following collection of examples apply to the foregoing formulæ, reference being made by the numbers in parentheses. Only a portion of the examples are worked out at length, but answers are given in every case.

Examples.

[1.] What would the interest at $4\frac{1}{2}$ per cent. upon £3653 for 7 years amount to?

Logarithm of 3653 = 3·562650
" 4·5 = 0·653213
" 7 = 0·845098

5·060961
2

3·060961 = Log of 1150·69.

∴ Answer is £1150 14*s.*

[2.] What would £364 put out at 6 per cent. compound interest yearly, amount to at the end of 23 years?

$$\begin{array}{rr}
\text{Log of ratio from table} = & 0.0253059 \\
 & 23 \\
\hline
 & 0.5420357 \\
\text{Logarithm of } 364 = & 2.561101 \\
\hline
 & 3.103137 = \text{Log of } 1268.05.
\end{array}$$

∴ Answer is £1268 1*s*.

[2.] What would £100 amount to at the end of 50 years, put out to annual compound interest at 5 per cent.? Ans. £1146 15*s*.

[3.] The sides of a triangle are respectively 564, 373, and 746, what is its area?

$$\begin{array}{lr}
\text{Log of } d = \tfrac{1}{2}(564+373+746) = & 2.925312 \\
\text{Log of } (d - a) = (842 - 564) = & 2.444045 \\
\text{Log of } (d - b) = (842 - 373) = & 2.671173 \\
\text{Log of } (d - c) = (842 - 746) = & 1.982271 \\
\hline
 & 2)10.022811 \\
\hline
 & 5.011405 = \text{Log of } 1026.61.
\end{array}$$

Therefore the area required is 1026·61.

[4.] What is the area of a square, the length of one side of which is 56·24 feet? Ans. 3162·94.

[5.] What is the area of a rectangle, the length of whose sides is 15·6 and 16·2? Ans. 252·62.

[6.] What is the area of a polygon of 12 sides, each of which is 5·06 feet in length?

$$\frac{90n - 180}{n} = 75^\circ$$

$$\begin{array}{lr}
\text{Logarithm of } l = 5.06 = & 0.704151 \\
 & 2 \\
\hline
 & 1.408302 \\
\text{Logarithmic tan of } 75^\circ = & 0.571948 \\
\text{Logarithm of } n = 12 = & 1.079181 \\
 & .397940 \\
\hline
 & 3.457371 \\
 & 1. \\
\hline
\text{Logarithm of area} = & 2.457371 = 286.663.
\end{array}$$

[6.] What is the area of an octagonal room, each side of which is 5 feet? Ans. 120·71.

[7, 8, and 9.] What is the circumference and area of a circle whose diameter is 21·72 feet?

$$
\begin{array}{rl}
\text{Logarithm of } d = 21\cdot72 = & 1\cdot336860 \\
 & \cdot497150 \\
\hline
\text{Logarithm of circumference} = & 1\cdot834010 = 68\cdot236. \\
 & 2 \\
\hline
 & 3\cdot668020 \\
 & 0\cdot900790 \\
\hline
 & 4\cdot568810 \\
 & 2\cdot \\
\hline
\text{Logarithm of area} = & 2\cdot568810 = 370\cdot52.
\end{array}
$$

[7, 8, and 9.] What is the diameter and circumference of a circle whose area is 562 square feet?

Ans. Circumference is 84·0376 feet, and diameter is 26·75 feet.

[10.] What is the length of an arc of 73° of a circle, whose radius is 34·72 feet? Ans. 44·237 feet.

[11.] What is the area of a sector of a circle whose radius is 26 feet, and whose sides include an angle of 42°? Ans. 247·58 feet.

[12.] What is the area of a parabola whose abscissa is 5·32, and the corresponding ordinate 4·13?

$$
\begin{array}{rl}
\text{Log of } (2\,y_2) = 2 \times 4\cdot13 = & 0\cdot916980 \\
\text{Log of } x_2 \quad = 5\cdot32 \quad = & 0\cdot725912 \\
 & \cdot823909 \\
\hline
 & 2\cdot466801 \\
 & 1\cdot \\
\hline
 & 1\cdot466801 = 29\cdot295 = \text{area of parabola.}
\end{array}
$$

[13.] In a parabola an ordinate measured 5·17, and its corresponding abscissa 8·95, what will be the length of the ordinate whose abscissa is 10?

$$
\begin{array}{rl}
\text{Log } y_1 = \quad 5\cdot17 = & 0\cdot713491 \\
 & 2 \\
\hline
 & 1\cdot426982 \\
\text{Log } x_2 = 10\cdot00 = & 1\cdot000000 \\
\hline
 & 2\cdot426982 \\
\text{Log } x_1 = \quad 8\cdot95 = & 0\cdot951823 \\
\hline
 & 2)1\cdot475155 \\
\hline
\text{Logarithm of } y_2 = & 0\cdot737577 = 5\cdot4648 = \text{the ordinate required.}
\end{array}
$$

[14 and 15. What is the area and periphery of an ellipse whose conjugate diameter is 27 and its transverse diameter is 49?

Ans. Area is 1039·08; and periphery is 119·38.

[16.] In an ellipse whose two diameters are 51 and 38, what is the length of the ordinate corresponding with an abscissa of 20 feet?

Since $x_1 = 20$, $x_2 = 51 - 20 = 31$.

$$\begin{array}{lr}
\text{Log } x_1 = 20 = & 1·301030 \\
\text{Log } x_2 = 31 = & 1·491367 \\
\hline
& 2)2·792397 \\
\hline
& 1·396198 \\
\text{Log } c = 38 = & 1·579784 \\
\hline
& 2·975982 \\
\text{Log } t = 51 = & 1·707570 \\
\hline
\text{Log } y = & 1·268412
\end{array}$$

$= 18·553 =$ the ordinate required.

[17.] What is the cubic contents of a cylinder whose diameter is 2·75 feet, and its height 6 feet? Ans. 35·637.

[18.] What is the cubic content of a cone whose diameter is 3·5 feet, and its height 5·42 feet? Ans. 34·764.

[19 and 20.] What is the spherical surface and the solidity of a sphere whose diameter is 5·734 feet?

Ans. Surface is 163·46 feet; solidity is 98·712 feet.

[21 and 22.] What is the surface and solidity of a tetraëdron, one of whose lineal edges is 7·31 feet, of an octaëdron whose lineal edge is 3·17, and of a dodecaëdron whose lineal edge is 5·69?

Ans. Tetraëdron, surface is 146·69 feet; solidity is 46·036 feet.
Octaëdron, surface is 34·81 feet; solidity is 15·005 feet.
Dodecaëdron, surface is 668·43 feet; solidity is 1411·7 feet.

[23 and 24.] In a plane triangle two of its sides are 7·3 and 6·92, and the angle opposite the longer side is 74° 39′, what are the remaining angles and the length of the other side?

Then by [23]

$$\begin{array}{lr}
\text{Logarithmic sin of } 74° \; 39' = & 9·984224 \\
\text{Arithm. comp. of log of } 7·3 = & 9·136677 \\
\text{Logarithm of . . . } 6·92 = & 0·840106 \\
\hline
& 19·961007 \\
& 10· \\
\hline
\text{Log sine of angle op. other side} = & 9·961007
\end{array}$$

$= 66° \; 4' \; 56''$.

Then, since the three angles of a triangle are equal to 180° we have

$180° - (74° 39' + 66° 4' 56'') = 39° 16' 4''$ for the angle opposite the side yet to be found.

Then by [24]

Logarithm of . . . . . . 6·92	=	0·840106
Arith. comp. of log sin of 66° 4′ 56″	=	0·038993
Logarithmic sin of . 39 16 4	=	9·801366
		10·680465
		10·
Logarithm of side required . .	=	0·680465 = 4·791.

Ans. The three sides are 4·791, 6·92, and 7·3, and the three angles opposite to each respectively are 39° 16′ 4″, 66° 4′ 56″, and 74° 39′.

[25 and 3.] Two sides of a triangular piece of ground measure 81·10 and 105·75, and the angle included between them is 47° 52′, what is the length of the other side, and the area of the piece of ground?

Log (105·75 − 81·10)	=	1·391817
Arith. comp. of log (105·75 + 81·1) .	=	7·728507
Logarithmic tan of . . . . . 66° 4′	=	10·352778
		19·473102
		10·
Log tan of half the difference of the angles opposite the given sides		9·473102 = 16° 33′ 14″.

Arith. comp. of log cos of 16° 33′ 14″	=	0·018384
Logarithmic cosine of . 66 4 0	=	9·608177
Logarithm of . . . (105·75 + 81·1)	=	2·271493
		11·898054
		10·
Logarithm of third side	=	1·898054 = 79·08. Ans.

Ans. And the area by [3] is 38355 square feet.

[26.] In a plane triangle whose sides are 27·3, 54·5, and 62, what are the angles opposite those sides respectively?

Arithmetical comp. of log of 62 .	=	8·207608
Log of . . . . (27·3 + 54·5)	=	1·912753
Log of . . . . (54·5 − 27·3)	=	1·434569
		11·554930
		10·
Log of the difference of the segments of the base	=	1·554930 = 35·886.

Therefore the larger segment is 31 + 17·943 = 48·943, and the lesser segment is 31 − 17·943 = 13·057.

Then by [23]

$$\begin{array}{lr} \text{Logarithmic sin of} \quad . \quad 90^\circ = & 10\cdot000000 \\ \text{Arith. comp. of log of } 54\cdot5 = & 8\cdot263603 \\ \text{Logarithm of} \quad . \quad . \quad 48\cdot943 = & 1\cdot689691 \\ \hline & 19\cdot953294 \\ & 10\cdot \\ \hline \end{array}$$

$$\text{Log sine of angle opp. larger segment} = 9\cdot953294 = 63^\circ\ 54'\ 4''.$$

Then $90^\circ - 63^\circ\ 54'\ 4'' = 26^\circ\ 5'\ 56''$ = angle opposite side which measures 27·3. Again,

$$\begin{array}{lr} \text{Logarithmic sine of} \quad . \quad 90^\circ = & 10\cdot000000 \\ \text{Arith. comp. of log of } 27\cdot3 = & 8\cdot563837 \\ \text{Logarithm of} \quad . \quad . \quad 13\cdot057 = & 1\cdot115843 \\ \hline & 19\cdot679680 \\ & 10\cdot \\ \hline \end{array}$$

$$\text{Log sin of angle opp. lesser segment} = 9\cdot679680 = 28^\circ\ 34'\ 23''.$$

Then $90^\circ - 28^\circ\ 34'\ 23'' = 61^\circ\ 25'\ 37''$ = the angle opposite the side which measures 54·5; and $28^\circ\ 34'\ 23'' + 63^\circ\ 54'\ 4'' = 92^\circ\ 28'\ 27''$ = the angle opposite the longest side.

[27.] What is the length of the diagonal of a rectangle whose two sides are 34 and 53·2? Ans. 63·14.

[28.] A house is 47 feet in height, at what distance must the base of a ladder 53 feet long be placed from the house in order that the top of the ladder may just meet that of the house? Ans. 24·5 feet.

[30.] What is the vis vivâ of a railway train weighing 117 tons, and travelling at a rate of 33 miles per hour? Ans. 19,313,300,000.

[31.] A body having been falling freely by the action of gravity for 7·5 seconds, it is desired to know the space which it has fallen through.
Ans. 915·37 feet.

[32 and 33.] A body falls under the influence of gravity from a height of 427 feet, what time will it occupy and what will be its final velocity, neglecting the resistance of the air?
Ans. It will occupy 4·1866 seconds, and acquire a velocity of 165·995 feet per second.

[34.] What length of time will a pendulum 34·7 inches in length be in making one vibration? Ans. 0·9416 seconds.

[35.] A body weighing 53 lbs. is whirled round in a circle whose radius is 15 feet, with a velocity of 12·7 feet per second, what is the strain upon the rope by which it is constrained to move in the circle? Ans. 17·703 lbs.

[36.] What is the horizontal pressure at the crown of an arch whose radius of curvature is 147·52 feet, whose breadth is 35 feet, and the vertical weight on each square foot at the key-stone is 974 lbs.? Ans. 5,028,950 lbs.

[37.] In an iron bridge having a span of 212 feet, with a rise of 22·5 feet, the weight of half the arch is 998 tons, and the distance of its center of gravity from the springing is 43 feet, what is the horizontal thrust of the arch?
Ans. 1907·3 tons.

[38.] A retaining wall 37 feet in height supports a loose sandy soil, required the pressure which every foot in length of it has to sustain?
Ans. 21447 lbs.

[38.] What is the pressure against a sluice 20 feet wide, and having a depth of 7 feet water against it?
Ans. 30,012,360 lbs.

[39 and 40.] What resistance would a board whose area is 14·7 square feet experience in being moved through the air with a velocity of 17 feet per second, and what would be the resistance in water?
Ans. In air, 72·221 lbs.; in water, 4146·34 lbs.

[42.] What quantity of water will be discharged by a pipe 18 inches in diameter, 5371 feet long, and under a head of 75 feet?

$$\begin{array}{rr}
\text{Log of } d = 18 = & 1\cdot2552725 \\
 & 5 \\
\hline
 & 6\cdot2763625 \\
\text{Log of } h = 75 = & 1\cdot8750613 \\
 & 1\cdot3485000 \\
\hline
 & 9\cdot4999238 \\
\text{Log } (l + 4\cdot2d) = 5446\cdot6 = & 3\cdot7361255 \\
\hline
 & 2)5\cdot7637983 \\
\hline
\end{array}$$

$$\text{Log of quantity per minute} = 2\cdot8818991 = 761\cdot9.$$

Ans. 761·9 cubic feet per minute.

[44.] What head will be required to force 350 cubic feet of water per minute through a pipe 15·5 inches in diameter, and 3640 feet long?
Ans. 22·739 feet.

[45.] What is the velocity with which water will flow through a conduit, 15 feet wide at the surface, 4 feet deep, with the sides sloped at 1 to 1, and the inclination of the surface of the water in which is 6 inches per mile?
Ans. 1·383 feet per second.

[46.] What is the quantity of water flowing over a weir 127 feet long, when the surface of the river is 6 inches above the top of the weir?
Ans. 145·93 cubic feet per second.

[47.] What weights would be requisite to tear asunder rods 2 inches square, of cast iron, wrought iron, oak, and fir?
Ans. Cast iron, 71,680 lbs.; wrought iron, 235,810 lbs.; oak, 47,520 lbs.; fir, 38,000 lbs.

[48.] What weight will be necessary to crush a block of cast iron 3 inches square?
Ans. 969,750 lbs.

[52.] What weight will be required to break a hollow column with flat

ends, the length of which is 37 feet, its external diameter 12 inches, and its internal diameter 10 inches?

$$\text{Log of } D = 12 = 1\cdot079181$$
$$\underline{3\cdot6}$$
$$3\cdot8850516 = \text{log of } D^{3\cdot6} = 7674\cdot6.$$

$$\text{Log of } d = 10 = 1\cdot000000$$
$$\underline{3\cdot6}$$
$$3\cdot6000000 = \text{log of } d^{3\cdot6} = 3981\cdot1$$
$$D^{3\cdot6} - d^{3\cdot6} = 3693\cdot5$$

$$\text{Log } (D^{3\cdot6} - d^{3\cdot6}) = 3693\cdot5 = 3\cdot567438$$
$$c = 1\cdot173186$$
$$0\cdot473217$$
$$5\cdot213841$$
$$\text{Log } l = 37 = 1\cdot568202 \times 1\cdot7 = 2\cdot6659434$$
$$2\cdot547898 = \text{log of } 353\cdot1.$$

Therefore the answer is 353·1 tons.

[53.] A bar of cast iron 2 inches wide and 3 inches deep is laid upon supports 6 feet apart, what weight applied in the center would break it?

Ans. 6135 lbs.

[54.] What weight applied in the center will be required to break a girder of Professor Hodgkinson's form of section, in which the area of the bottom flange is 26 square inches, the depth 15 inches, and the distance between the supports 23 feet?

Ans. 82,273 lbs.

[55.] What deflexion will be produced in a bar of cast iron 2 inches wide, 3 inches in depth, and with a 6 feet bearing, by a weight of 2730 lbs. applied in the center?

Ans. ·256 inch.

APPENDIX.

Table of the Logarithms of every Prime Number from 2 to 1000.

Prime number.	Logarithm.	Prime number.	Logarithm.	Prime number.	Logarithm.	Prime number.	Logarithm.
2	3010300	191	2810334	439	6424645	709	8506462
3	4771213	193	2855573	443	6464037	719	8567289
5	6989700	197	2944662	449	6522463	727	8615344
7	8450980	199	2988531	457	6599162	733	8651040
11	0413927	211	3242825	461	6637009	739	8686444
13	1139434	223	3483049	463	6655810	743	8709888
17	2304489	227	3560259	467	6693169	751	8756399
19	2787536	229	3598355	479	6803355	757	8790959
23	3617278	233	3673559	487	6875290	761	8813847
29	4623980	239	3783979	491	6910815	769	8859263
31	4913617	241	3820170	499	6981005	773	8881795
37	5682017	251	3996737	503	7015680	787	8959747
41	6127839	257	4099331	509	7067178	797	9014583
43	6334685	263	4199557	521	7168377	809	9079485
47	6720979	269	4297523	523	7185017	811	9090209
53	7242759	271	4329693	541	7331973	821	9143432
59	7708520	277	4424798	547	7379873	823	9153998
61	7853298	281	4487063	557	7458552	827	9175055
67	8260748	283	4517864	563	7505084	829	9185545
71	8512583	293	4668676	569	7551123	839	9237620
73	8633229	307	4871384	571	7566361	853	9309490
79	8976271	311	4927604	577	7611758	857	9329808
83	9190781	313	4955443	587	7686381	859	9339932
89	9493900	317	5010593	593	7730547	863	9360108
97	9867717	331	5198280	599	7774268	877	9429996
101	0043214	337	5276299	601	7788745	881	9449759
103	0128372	347	5403295	607	7831887	883	9459607
107	0293838	349	5428254	613	7874605	887	9479236
109	0374265	353	5477747	617	7902852	907	9576073
113	0530784	359	5550944	619	7916906	911	9595184
127	1038037	367	5646661	631	8000294	919	9633155
131	1172713	373	5717088	641	8068580	929	9680157
137	1367206	379	5786392	643	8082110	937	9717396
139	1430148	383	5831988	647	8109043	941	9735896
149	1731863	389	5899496	653	8149132	947	9763500
151	1789769	397	5987905	659	8188854	953	9790929
157	1958997	401	6031444	661	8202015	967	9854265
163	2121876	409	6117233	673	8280151	971	9872192
167	2227165	419	6222140	677	8305887	977	9898946
173	2380461	421	6242821	683	8344207	983	9925535
179	2528530	431	6344773	691	8394780	991	9960737
181	2576786	433	6364879	701	8457180	997	9986952

Prime numbers are those which are not divisible by any other number, or which cannot be resolved into factors; thus 233 is a prime number, because it cannot be divided by any number without leaving a remainder, while 234 is not a prime number, it being divisible by 2 and other numbers. The logarithms of any number which is not a prime number may be readily found by adding together the logarithms of the several prime factors by the multiplication of which the number is produced. Thus the number 234 is produced by the multiplication of 2, 3, 3 and 13 (all prime numbers,) and the logarithms of those numbers being taken from the table and added together, the sum will be the logarithm of 234. For example—

$$\begin{aligned} \text{Log of } 2 &= 0{\cdot}3010300 \\ \text{,, } 3 &= 0{\cdot}4771213 \\ \text{,, } 3 &= 0{\cdot}4771213 \\ \text{,, } 13 &= 1{\cdot}1139434 \\ \therefore \text{Log of } 234 &= \underline{2{\cdot}3692160} \end{aligned}$$

Again, the number 578 is composed of the prime factors 2, 17, and 17; then

$$\begin{aligned} \text{Log of } 2 &= 0{\cdot}3010300 \\ \text{,, } 17 &= 1{\cdot}2304489 \\ \text{,, } 17 &= 1{\cdot}2304489 \\ \therefore \text{Log of } 578 &= \underline{2{\cdot}7619278} \end{aligned}$$

In this manner we are enabled by the foregoing table to find the logarithm (true to at least 6 figures) of any number which may be given, whether prime or otherwise; for if prime its logarithm will be found at once in the table, but if not prime its logarithm will then be found by taking the sum of the logarithms of its prime factors, as explained above.

Table by the aid of which the number answering to any logarithm can be found to six places.

		1	10	100	1000	10000	100000
1	0000000	0413927	043214	04341	0434	043	04
2	3010300	0791812	086002	08677	0869	087	09
3	4771213	1139434	128372	13009	1303	130	13
4	6020600	1461280	170333	17337	1737	174	17
5	6989700	1760913	211893	21661	2171	217	22
6	7781513	2041200	253059	25980	2605	261	26
7	8450980	2304489	293838	30295	3039	304	30
8	9030900	2552725	334238	34605	3473	347	35
9	9542425	2787536	374265	38912	3907	391	39

In the above table the arguments are natural numbers, and the resultants their logarithms. The first figures of the arguments are found in the top horizontal line, and the final or unit's figure of the same in the extreme left-hand column; the logarithm is found at the place of intersection, that is, on the same line with the final figure, and in the same column as the other figures of the natural number. In the five last columns only the final *significant* figures of the mantissa of the logarithms will be found in the table; as many cyphers must be added to the left of the figures given as are necessary to make up seven figures. Thus at the top of the fifth column we have 100, and on the fourth line we have 4, then the figures found at the place of intersection are 17337, to which adding two cyphers on the left hand to make up the seven figures, we have ·0017337, which is the mantissa of the logarithm of 1004.

The manner of using the table is as follows:—Having given a logarithm of which it is desired to know the corresponding number, look among the resultants in the table for the next less number to the mantissa of the given logarithm, and write down the natural number corresponding with the logarithm taken from the table, subtract this logarithm from the mantissa given, and again look among the resultants in the table for the next less number to the remainder, noting the number among the arguments answering to it; then subtract the resultant from the remainder, and look again for the next less resultant to this remainder, and thus proceed until the given logarithm has been exhausted, that is, until no remainder is left, each time noting the natural numbers cor-

responding to the logarithms taken from the table. These numbers being then multiplied together, the product will be the natural number corresponding to the logarithm originally given. These numbers have been so arranged that their multiplication may be very readily performed.

Example.—Of what number is 3·3574202 the logarithm?

Given logarithm	=	·3574202	
Next less log in table . .	=	·3010300	= log of 2
1st remainder	=	563902	
Next less logarithm . . .	=	413927	= log of 11
2nd remainder	=	149975	
Next less logarithm . . .	=	128372	= log of 103
3rd remainder	=	21603	
Next less logarithm . . .	=	17337	= log of 1004
4th remainder	=	4266	
Next less logarithm . . .	=	3907	= log of 10009
5th remainder	=	359	
Next less logarithm . .	=	347	= log of 100008
6th remainder	=	12	
Nearest logarithm	=	13	= log of 1000003

Then $2 \times 11 \times 103 = 2266$, which has to be next multiplied by 1004, or by 1000 and by 4, thus

$$
\begin{array}{r}
2266000 = 2266 \times 1000 \\
9064 = 2266 \times 4 \\
\hline
2275064 \quad\quad\quad\quad\quad\quad
\end{array}
$$

This again has to be multiplied by 10009,

$$
\begin{array}{r}
22750640000 \\
20475576 \\
\hline
22771115576
\end{array}
$$

This has again to be multiplied by 100008, but we need not retain more than 8 figures, and the remainder to the right may be cut off, and any figures in the multiplication by 8 which would fall under any of the figures so cut off may be omitted; to know how many figures thus to omit, point off as many figures from the right as there are figures before the number by which you are about to multiply, and perform the multiplication only upon the remaining figures, taking care, however, to carry to the multiplication of the first number whatever would have been carried from that of the last figure cut off. Thus, in the example, there being *five* figures before 8, the number by which we are going to multiply, we point off the *five* right hand figures, and only multiply 227 by 8; we add in, however, 6 carried from the multiplication of the 7 cut off.

```
227.71115
     1822
---------
22.772937
       68
---------
 22773005
```

The last multiplication is by 1000003, and the answer is true to seven places, the real number being 2277·3. In the example above, as the figures to be multiplied by 1000003 are not affected by the addition of 1822, this need not have been performed until afterwards, as below.

```
227.71115
     1822
       68
---------
 22773005
```

What is the number corresponding to the logarithm 4·8551071?

8551071	
8450980 = log of 7	
100091	
86002 = log of 102	
14089	
13009 = log of 1003	
1080	
869 = log of 10002	
211	
174 = log of 100004	
37 = log of 1000009	

7 × 102 =	714000
	2142
	71,6,1,4200
	14323
	2865
	643
	71632031

The number is 71632.

What is the number whose logarithm is 2·6103833?

6103833
6020600 = log of 4
83233
43214 = log of 101
40019
38912 = log of 1009
1107
869 = log of 10002
238
217 = log of 100005
21 = log of 1000005

101
4
404000
3636
40,7,6,3600
8153
2038
204
40773995

The answer is 407·74.

What is the number whose logarithm is 3·7797587?

7797587	1003
7781513 = log of 6	6
16074	60,18,0000
13009 = log of 1003	42126
	361
3065	
3039 = log of 10007	60222487
26 = log of 1000006	

The number required is 6022·248.